Opt Quant Electron (2016) 48:434
DOI 10.1007/s11082-016-0700-1

Dynamics of entropy and quantum statistical properties of the field in the interaction of a single two-level atom with a superposition of nonlinear coherent states in the framework of f-deformed Jaynes–Cummings model

Omid Abbasi[1] · **Akbar Jafari**[1]

Received: 9 March 2016 / Accepted: 12 August 2016

Abstract Based on the f-deformed Jaynes–Cummings model, we investigate the interaction of a single two-level atom with a general superposition of nonlinear coherent states (NCSs) with intensity-dependent coupling in a Kerr-like medium. We propose a solution of this model based on the density operator method. As an especial case the interaction of the atom with an even nonlinear coherent state (ENCS) is studied. By considering the effect of the field intensity and the detuning parameter, we compare such a system with the corresponding linear system which has been already studied in Vidiella-Baranco et al. J. Mod. Opt. 39:1441, (1992) and Gerry and Hach Phys. Lett. A 179:1, (1993). The atomic inversion behaviors of the linear and nonlinear interactions are found to be completely different. However, the most significant features of these interactions is that the field in nonlinear system exhibits sub-Poissonian statistics while the linear one do not show such a nonclassical property. In order to evaluate the degree of entanglement between the atom and the field, we investigate the dynamics of the entanglement by studying the field entropy in both linear and nonlinear cases. Finally studying the field evolutions using the Q-function has shown that the initial fields develop into multiple Schrödinger cat-like states. Especially we found that for the exact resonant case at successive collapses of atomic inversion, the initial field components split into a superposition of the four-component NCSs $\pi/2$ out of phase which may be potentially considered as the source of the new notion of "four-photon nonlinear coherent states".

Keywords Superposition of nonlinear coherent states · f-Deformed Jaynes–Cummings Model · Quantum entanglement · Nonclassical properties

✉ Akbar Jafari
a.jafari@urmia.ac.ir

Omid Abbasi
o.abbasi@urmia.ac.ir

[1] Atomic and Molecular Group, Faculty of Science, Urmia University, Urmia, Iran

Published online: 24 August 2016

1 Introduction

"Quantum Superposition" is a fundamental principle in quantum mechanics and has extensive applications in the field of quantum optics. Specifically superpositions of various states of light have been studied during the last four decades. The notion of standard coherent states (CSs) was introduced by Glauber (1963) as eigenstates of the annihilation operator of the harmonic oscillator. Since then, superpositions of these states have been studied because it has been realized that the quantum interference effects may give rise to various non-classical properties like squeezing, higher-order squeezing as well as sub-Poissonian statistics and oscillations in photon number distribution. Particularly, the "*Schrödinger cat states*" which are defined as the quantum superposition of two quasi-classical macroscopically distinguishable states of light have numerous theoretical and practical applications in many quantum information processing (Leek 2013; Velastakis et al. 2013). A general optical cat state is defined as:

$$|\psi\rangle = \mathcal{N}(|\alpha\rangle + e^{i\phi}|-\alpha\rangle), \tag{1}$$

where ϕ is the relative phase and $\mathcal{N}$ is the normalization factor and the CSs $|\alpha\rangle$ and $|-\alpha\rangle$ can be interpreted respectively as "alive" and "dead" states. The concepts of "even" and "odd" CSs (Dodonov et al. 1974), which are respectively symmetric and anti-symmetric superpositions of CSs, have considerable non-classical features. The even (odd) CSs show squeezing (antibunching) but no antibunching (no squeezing) effects. Such states can be recovered as special cases of (1) when $\phi = 0$ (even CSs) and $\phi = \pi$ (odd CSs); therefore they are often referred to as even (odd) cat states. The authors of Zeng et al. (2007) introduced a particular superposition of $\pi/2$ out of phase CSs. They constructed the superposed states in the form of $\frac{\mathcal{N}}{\sqrt{2}}(|\alpha\rangle + e^{i\phi}|i\alpha\rangle)$ and showed that the special nonclassicality was found where the relative phase factor has a specific relation with the average photon number. In the meantime when Schleish et al. (1991) were working on the effect of phase difference on the superposition of states $|\alpha e^{i\theta/2}\rangle$ and $|\alpha e^{-i\theta/2}\rangle$ with $\alpha \in \mathbb{R}$, it was shown that the squeezing and sub-Poissonian statistics can simultaneously occur in the limit of large average photon numbers as well as oscillatory photon statistics. Alongside these theoretical investigations, a considerable amount of theoretical and experimental efforts have been spent to develop several schemes which could generate quantum superposition states via numerous nonlinear processes (Gerry 1992; Ourjoumtsev et al. 2007; Lee et al. 2012).

The concept of nonlinear coherent states (NCSs) or f-deformed CSs (Matos Filho and Vogel 1996; Man'ko et al. 1997; Recamier et al. 2008) has been the subject of some important debates in quantum optics, especially because of their non-classical properties. The approach in Man'ko et al. (1997) is based on the algebraic generalization of standard CSs by using f-deformed annihilation operator $\hat{A} = \hat{a}f(\hat{n})$ where $f(\hat{n})$ is an operator-valued function of number operator $\hat{n} = \hat{a}^{\dagger}\hat{a}$ and characterizes the nonlinearity nature of the system. In the last fifteen years, much attention has been paid to the superposition of NCSs (Abbasi and Tavassoly 2010; Sivakumar 2000b); for example the extension of even (odd) CSs to the even (odd) NCSs (Mancini 1997; Sivakumar 1998; Meng et al. 2007) as well as a theoretical scheme for generating these states (Sivakumar 2000a) have been widely investigated. The well-known expression for even (+) and odd (−) NCSs is given by:

$$|\alpha, f\rangle_{\pm} = N_{\pm}(|\alpha, f\rangle \pm |-\alpha, f\rangle). \tag{2}$$

It is found that these states, which can be interpreted as *Schrödinger cat states* for deformed fields, might possess some of the previously mentioned nonclassical properties depending on the form of nonlinearity. Another important superposition of NCSs is a generalization of the introduced states in Zeng et al. (2007), using the procedure proposed by one of us in Abbasi and Tavassoly (2009), where the superposition of two sets of $\pi/2$ out of phase NCSs is constructed in the following form:

$$|\psi_f\rangle = \frac{N_s}{\sqrt{2}}(|\alpha, f\rangle + e^{i\phi}|i\alpha, f\rangle). \tag{3}$$

In the above formula ϕ and N_s are an arbitrary phase factor and the normalization factor, respectively. It was shown that depending on the form of $f(\hat{n})$, this kind of superimposed states can demonstrate some obvious nonclassical properties such as oscillatory photon count probability, sub-Poissonian statistics, amplitude-squared squeezing and the negativity of Wigner distribution function. It is worth mentioning that unlike the even and odd NCSs, which do not belong to the category of nonlinear CSs, this specific superposition would result in nonclassical states which are nonlinear CSs. This means that these states can in principle be obtained as the eigenstate of the generalized annihilation operator (Abbasi and Tavassoly 2009).

On the other hand interaction of a single-mode of the quantized electromagnetic field with a two-level atom is a challenging problem in atom-radiation field studies. The standard model for describing this quantum system was introduced by Jaynes and Cummings (1963). The Jaynes–Cummings model (JCM) is an exactly solvable model and can be used for considering the interaction of various states of the cavity fields such as superposition of CSs with two-level atoms. In fact the sensitivity of the system to the initial field photon statistics is the motivation of studying the interaction of the atom with various states of the field that indicate nonclassical properties, such as the superpositions of CSs (Gerry and Hach 1993; Vidiella-Baranco et al. 1992; Vidiella-Baranco and Moya-Cessa 1995). For example the interaction of a two-level atom with a quantized cavity field in an even coherent state is investigated in Gerry and Hach (1993). Also in Vidiella-Baranco et al. (1992), the authors compare the behavior of a system of interacting atom and fields where a superposition of two CSs (even CSs as the special case) and a statistical mixture of CSs is considered as the initial field. On the other hand the influence of the relative phase factor of the initial field [see (1)] on the purity of the field was studied in Vidiella-Baranco and Moya-Cessa (1995). In Vidiella-Baranco and Moya-Cessa (1995), the same authors presented a superposition of the squeezed states and their interactions with two-level atoms. Due to the potential of the standard JCM in developing theoretical results which agree with experimental observations, this standard model has been generalized in different ways in the past fifty years. For instance in Tavis-Cummings model the interaction between the cavity field and a group of two level atoms has been investigated (Tavis and Cummings 1968). In addition, when the interaction between the field and atoms is no longer linear in the field variables, a specific form of intensity-dependent coupling is considered (Buck and Sukumar 1981; Buzek 1989; Hekmatara and Tavassoly 2014). Also when the atom is surrounded by a nonlinear Kerr-like medium, the medium is modeled as an anharmonic oscillator to describe the excitation of the electromagnetic field mode (Agarwal and Puri 1989). As another generalization of the standard model, the model has been extended to multi-level atoms and multi-mode electromagnetic fields in different studies (Zait 2003; Faghihi and Tavassoly

2012; Skrypnyk 2013) as well as multi photon JCM (Abdel-Khalek et al. 2011; Panahi and Asghari Rad 2013; Abdel-Khalek et al. 2015).

Lately, a special type of the generalized Jaynes–Cummings Model has been introduced in de los Santos-Sanches and Recamier (2012) by using the framework of an f-deformed oscillator (Man'ko et al. 1997). In fact the authors presented a new type of nonlinear Jaynes–Cummings model in which a specific form of nonlinearity operator valued function $f(\hat{n})$ makes it possible to achieve two of the above generalizations simultaneously (atom-field intensity-dependent coupling and Kerr-like nonlinearity). In addition this model allows one to study the interaction of NCSs and their evolution with the atom because the medium has been modeled by a deformed oscillator instead of a standard harmonic oscillator. It is notable that such a model as a theoretical generalized atom-field interaction model can be applied to real problems with nonlinear interactions in the field of nonlinear optics. For instance in Walentowitz and Vogel (1997), Matos Filho and Vogel (1998) it has been shown that Raman-type laser excitation of a trapped atom allows one to realize counterpart of nonlinear optics phenomena such as Kerr nonlinearity, parametric interaction and several types of nonlinear wave mixings, therefore such a model can be realized in experiments.

As the main goal of our paper we want to investigate the interaction of a single two-level atom inside a lossless cavity in presence of nonlinear Kerr-like medium with single-mode of electromagnetic field which initially prepared in a superposition of NCSs. Although dissipation plays a crucial role in our problem, in order to extract the characteristics of our mechanism, we first need to investigate the ideal case in which there would not be any dissipation present. In Sect. 2.1.we introduce a general superposition of nonlinear CSs, in which we deal with two arbitrary phase factors: a relative phase factor ϕ between the initial components and the phase difference θ between the nonlinear CSs amplitudes. In order to describe the interactions in this system, in Sect. 2.2 we present a solution of the f-deformed JCM based on the density operator method. This allows us to consider the atom and the cavity field to be initially in arbitrary states, for instance atom might be in a superposition of the ground and the excited states and the field might be in a pure state as well as a statistical mixture of NCSs. In Sect. 3 we study the evolution of atomic inversion in the interaction system. The statistical properties of the field are investigated using the Mandel q-parameter in Sect. 4.1. In Sect. 4.2 the quantum entanglement due to the atom-field interaction is evaluated via the time evolution of the field entropy. The evolution of the field quantum state has been analyzed using the Q-function in Sect. 4.3. It should be noted that although our proposed approach is quite general, but for our numerical analyses we choose especial values for relative phase factor and phase difference: $\phi = 0, \theta = \pi$, so in fact we investigate the interaction of a single two-level atom with an even NCSs as the initial field inside a lossless optical cavity. Due to the existence of a critical point in generalized Rabi frequency of the f-deformed JCM as a function of the photon number n, during our calculations we study the effects of the initial field intensity as well as detuning parameter on the evolution of atomic inversion, statistical properties of the field, dynamics of entropy and the evolution of the field quantum state. Finally if we choose $f(\hat{n}) = 1$, our presented model recovers the results of Gerry and Hach (1993), Vidiella-Baranco et al. (1992), in which we would have the superposition of standard CSs interacting with a single two-level atom based on the standard JCM Hamiltonian. Therefore we would be able to compare the nonlinear and linear problems to have an estimate of the nonlinear effects.

2 Interaction between a two-level atom with superposition of nonlinear coherent states

2.1 General superposition of nonlinear coherent states

The notion of f-deformed coherent states (f-deformed CSs) or NCSs is based on the deformation of standard annihilation and creation operators with a nonlinearity function of the usual number operator $\hat{n} = \hat{a}^\dagger \hat{a}$ according to the relations (Man'ko et al. 1997):

$$\hat{A} = \hat{a} f(\hat{n}) = f(\hat{n}+1)\hat{a} \tag{4}$$

$$\hat{A}^\dagger = f^\dagger(\hat{n})\hat{a}^\dagger = \hat{a}^\dagger f^\dagger(\hat{n}+1), \tag{5}$$

where $\hat{a}$ and $\hat{a}^\dagger$ are bosonic annihilation and creation operators, respectively and the commutators between $\hat{A}$ and $\hat{A}^\dagger$ read as:

$$[\hat{A}, \hat{A}^\dagger] = (\hat{n}+1)f^\dagger(\hat{n}+1)f(\hat{n}+1) - \hat{n}f^\dagger(\hat{n})f(\hat{n}). \tag{6}$$

Now the NCSs are defined as the eigenstates of the generalized annihilation operator in the following eigenvalue equation:

$$\hat{A}|\alpha, f\rangle = \alpha|\alpha, f\rangle, \qquad \alpha \in \mathbb{C}. \tag{7}$$

The explicit form of NCSs in Fock space representation is:

$$|\alpha, f\rangle = N_f \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}\ [f(n)]!}\ |n\rangle, \tag{8}$$

where N_f is some appropriate normalization factor determined from $\langle \alpha, f|\alpha, f\rangle = 1$ by the convention $[f(n)]! \doteq f(1)f(2)\ldots f(n), \quad [f(0)]! \doteq 1.$ and is determined as:

$$N_f = \left(\sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!\ [f(n)f^\dagger(n)]!} \right)^{-\frac{1}{2}}. \tag{9}$$

In this section we introduce the superposition of NCSs with two arbitrary phase differences. The first phase difference ϕ is between the two components of the superposed state and the second one θ is the phase difference between the components amplitudes. In general, this superposition is given by:

$$|\psi_s\rangle = N_s(|\alpha, f\rangle + e^{i\phi}|\alpha e^{i\theta}, f\rangle), \tag{10}$$

where the normalization factor N_s can be easily determined by the condition $\langle \psi_s|\psi_s\rangle = 1$ as follow:

$$N_s = \left(2 + 2|N_f|^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}\cos(\phi + n\theta)}{n!\ [f(n)f^\dagger(n)]!} \right)^{-\frac{1}{2}}. \tag{11}$$

As an important quantity we can calculate the photon number distribution of (10) as:

$$P(n) = |\langle n|\psi_s\rangle|^2 = \frac{4|N_s|^2|N_f|^2|\alpha|^{2n}}{n!\,[f(n)f^\dagger(n)]!}\cos^2\left(\frac{\phi+n\theta}{2}\right). \tag{12}$$

Note that due to the existence of the term $[\cos^2(\frac{\phi+n\theta}{2})]$ in (12), oscillatory photon count probability as an important nonclassical property is the essential feature of the superposed states in (10). As a special case by setting $f(n) = 1$ in (10), one can readily obtain the general superposition of the standard CSs:

$$|\psi_s\rangle_{linear} = N_s\big(|\alpha\rangle + e^{i\phi}|\alpha e^{i\theta}\rangle\big). \tag{13}$$

It can be easily seen that the prominent property of the form (10) is that by substituting ϕ and θ with special values, one can obtain particular superpositions of NCSs presented before. For example setting $\phi = 0(\pi)$ and $\theta = \pi$ in Eq. (10), we would achieve the well-known even ($\phi = 0$) and odd ($\phi = \pi$) NCSs. Also for any optional ϕ in (10) by substituting $\theta = \pi/2$, the superposed states introduced by one of us in Abbasi and Tavassoly (2009) can be recovered.

2.2 f-Deformed JCM

For studying the interaction between superposed states introduced in (10) and a single two-level atom, we present a solution of the f-deformed JCM based on the density operator method. It should be mentioned that the f-deformed JCM which was introduced for first time in de los Santos-Sanches and Recamier (2012), can be used for describing the interaction between NCSs and a single two-level atom. In fact, this model is a generalization of the standard JCM including the two generalizations mentioned in the introduction, i.e., intensity-dependent atom-field coupling and the assumption of an additional Kerr-like medium inside the cavity. On the other hand, using the generalized creation and annihilation operators defined in (4) and (5) for describing the cavity field allows us to consider f-deformed CSs as the initial field of the cavity.

We start with the standard JCM which is one of the exactly solvable models for describing the problem of a single two-level atom interacting with single-mode electromagnetic field inside a lossless cavity. The total Hamiltonian for this system in dipole and rotating wave approximations can be written as (Jaynes and Cummings 1963; Shore and Knight 1993):

$$\hat{H} = \hbar\omega_0\hat{a}^\dagger\hat{a} + \frac{1}{2}\hbar\omega\hat{\sigma}_3 + \hbar\lambda(\hat{\sigma}_+\hat{a} + \hat{a}^\dagger\hat{\sigma}_-), \tag{14}$$

where the first two terms are the energy operators of the free field and the atom respectively in the absence of interaction and the third one describes the interaction between the atom and the radiation field. The atomic operators are $\hat{\sigma}_3 = |e\rangle\langle e| - |g\rangle\langle g|, \hat{\sigma}_+ = |e\rangle\langle g|$ and $\hat{\sigma}_- = |g\rangle\langle e|$ with the commutation relation $[\hat{\sigma}_3, \hat{\sigma}_\pm] = \pm 2\hat{\sigma}_\pm$ and $[\hat{\sigma}_+, \hat{\sigma}_-] = \hat{\sigma}_3$. We suppose that the excited and the ground states of the atom being $|e\rangle$ and $|g\rangle$ respectively. Also ω_0 and ω are the field and the atomic transition frequencies. Please note that the constant atom-field coupling is denoted by λ. Now by considering the generalized annihilation and creation operators, the f-deformed JCM Hamiltonian can be written as follows (de los Santos-Sanches and Recamier 2012):

$$\hat{\mathcal{H}} = \hbar\omega_0\hat{A}^\dagger\hat{A} + \frac{1}{2}\hbar\omega\hat{\sigma}_3 + \hbar\lambda(\hat{\sigma}_+\hat{A} + \hat{A}^\dagger\hat{\sigma}_-), \tag{15}$$

where the first term is the Hamiltonian of the deformed free field and may be considered as the "*normal-ordered*" of the Man'ko et al Hamiltonian $\frac{\hbar\omega_0}{2}(\hat{A}^\dagger\hat{A}+\hat{A}\hat{A}^\dagger)$ (Roknizadeh and Tavassoly 2004; Man'ko et al. 2010). The second term is still the energy operator of the atom but the atom-field coupling of the interaction part of the Hamiltonian is now intensity-dependent due to the existence of the nonlinearity function $f(\hat{n})$. It is clear that the main difference between (15) and (14) which describe distinct physical systems is in the nonlinearity nature of $f(\hat{n})$. This means that the physical properties of the interaction system, for instance the nonclassical properties of the field inside the cavity, described by (15) can be controlled by the nonlinearity function and consequently the appropriate choice of $f(\hat{n})$ may lead to special nonlinearity of the atom-field interaction. For reaching specific physical results, in this paper the following real nonlinearity function is used (de los Santos-Sanches and Recamier 2012; Sivakumar 2004; Roman-Ancheyta et al. 2014):

$$f(\hat{n}) = \sqrt{1-\frac{\chi}{\omega_0}(1-\hat{n}^{k-1})}, \tag{16}$$

where χ denotes the dispersive part of the third-order nonlinearity of the Kerr medium with $0 \le \chi \ll \omega_0$ and k can take the discrete values 1, 2, 3…. By Substituting (16) in (15), the explicit form of the f-deformed JCM Hamiltonian is obtained as:

$$\begin{aligned}\hat{\mathcal{H}}_f = \hbar\nu\hat{n} + \hbar\chi\hat{n}^k + \frac{1}{2}\hbar\omega\hat{\sigma}_3 + \hbar\lambda\Bigg(\hat{\sigma}_+\hat{a}\sqrt{1-\frac{\chi}{\omega_0}(1-\hat{n}^{k-1})}\\ + \sqrt{1-\frac{\chi}{\omega_0}(1-\hat{n}^{k-1})}\hat{a}^\dagger\hat{\sigma}_-\Bigg),\end{aligned} \tag{17}$$

where $\nu = \omega_0 - \chi$. As the first physical result of using deformation function (16), one can easily find that the first two terms of (17) for $k = 2$, would construct the Kerr Hamiltonian as (Mandel and Wolf 1995):

$$\hat{H}_{Kerr} = \hbar\omega_0\hat{a}^\dagger\hat{a} + \hbar\chi\hat{a}^{\dagger 2}\hat{a}^2. \tag{18}$$

where the optical properties of the Kerr medium are described with the second term. It is worth mentioning that the first two terms of (17) can be considered as a special case of the anharmonic oscillator Hamiltonian:

$$\hat{H} = \omega_0\hat{n} + \chi\hat{n}^k, \tag{19}$$

which has been studied by Yurke and Stoler in Yurke and Stoler (1986). They showed that under the evolution of the Hamiltonian (19) an initial standard coherent state $|\alpha\rangle$ will evolve into a coherent superposition of a finite number of CSs that are macroscopically distinguishable when $|\alpha|$ is large enough. Particularly when k in (19) is even, at $t = \pi/2\chi$, the initial coherent state has evolved into a Schrödinger cat state with two components which are π out of phase i.e. $|\alpha\rangle$ and $|-\alpha\rangle$. Consequently the f-deformed Hamiltonian (15) with the deformation function (16) and special case $k = 2$ describes a system in which an atom is surrounded by a Kerr-like medium. On the other side the existence of such nonlinearity function in the interaction part of the Hamiltonian results in a special intensity-dependent coupling because we deal with $\lambda f(\hat{n})$ instead of the atom-field constant coupling λ of the standard JCM. But as the main physical consequent in this approach using the nonlinearity function (16) to construct the f-deformed annihilation and creation operators allows one to consider the field f-deformed CSs (NCSs) and their evolutions.

In addition such a nonlinearity function has been used in some works to describe the CSs of non-harmonic oscillators in the term of Morse potential. For example in Recamier and Jauregui (2003) the authors use such a nonlinearity function to construct the even and odd Morse-like CSs and evaluate the temporal evolution of the position operator and its dispersion as a function of time when the states evolve under a nonlinear Morse Hamiltonian. Therefore one may conclude that the physical system described by the f-deformed JCM can characterize the interaction of a single two-level atom with the Morse potential in a lossless optical cavity where the cavity field is considered to be the related nonlinear CSs or their superpositions. Hereafter we would only consider the case $k = 2$.

One might be interested in isolating intensity-dependent nonlinearity effects and the Kerr like medium ones. However since such nonlinearities are described by the same parameter χ in the f-deformed JCM, it is clear that this kind of nonlinear JCM does not provide the necessary tools for a separate study of the influences of these nonlinearities. For differentiating the effects of these nonlinearities, one is required to correspond different parameters to them. A straightforward change of parameters which could due the trick would be as follows:

$$\hat{\mathcal{H}}_{NJCM} = \hbar\omega_0 \hat{a}^\dagger \hat{a} + \hbar\chi \hat{a}^{\dagger 2} \hat{a}^2 + \frac{1}{2}\hbar\omega\hat{\sigma}_3 + \hbar\lambda(\hat{\sigma}_+ \hat{R} + \hat{R}^\dagger \hat{\sigma}_-), \tag{20}$$

in which $\hat{R} = \hat{a} g(\hat{n})$ and $g(\hat{n})$ (in general different from $f(\hat{n})$) describes the intensity-dependent coupling between the atom and the field. It is evident that there is no prohibition to use the NCSs and their superpositions as the initial cavity field in a system described by (20) because they all are just superpositions of number states. Moreover, since the first two terms of (20) can be expressed as the deformed free field Hamiltonian $\hbar\omega_0 \hat{A}^\dagger \hat{A}$ [with nonlinearity function (16)], the initial cavity field can be generated by applying a deformed displacement operator $\hat{D}(\alpha) = \exp(\alpha\hat{A}^\dagger - \alpha^* \hat{A})$ on the vacuum state (Roman-Ancheyta et al. 2011).

There are several ways to solve the JCM. For instance in de los Santos-Sanches and Recamier (2012), solving the time-dependent Schrödinger equation has been followed. In this study we present a solution based on the density operator method. It is a well-known fact that the density operator is the quantum-mechanical approach for describing a quantum system which in general is in a mixed state. Therefore in our problem a solution based on the density operator method makes it possible to easily describe the interaction system in which the field can be initially in a mixed state as well as in a pure state like what we follow in this paper. Also the atom might be in ground state or in excited state or in any superposition of these states. The density operator for the atom-field system follows a unitary time evolution generated by the time evolution operator $U(t)$, and can be written as:

$$\hat{\rho}(t) = \hat{U}(t)\hat{\rho}(0)\hat{U}^\dagger(t) \tag{21}$$

where the $\hat{\rho}(t)$ is the density operator for the combined atom-field system and we suppose that at time $t = 0$ the atom is in its excited state and that the atom and the field are initially uncorrelated. Therefore the initial density operator in the atomic basis can be written as (Shore and Knight 1993):

$$\hat{\rho}(0) = \hat{\rho}_A(0) \otimes \hat{\rho}_F(0) = \begin{pmatrix} \hat{\rho}_F(0) & 0 \\ 0 & 0 \end{pmatrix}, \tag{22}$$

where $\hat{\rho}_{A(F)}$ are the reduced density operators and the subscript $A(F)$ denotes the atomic (field) subsystem. Now considering the Hamiltonian (15) with the nonlinearity function (16) and using a method similar to the procedure in Crnugelj et al. (1994), we find the time evolution operator:

$$\hat{U}(t) = \begin{pmatrix} \cos\hat{\Phi}_n t - i\frac{\hat{\Lambda}_n}{2\hat{\Phi}_n}\sin\hat{\Phi}_n t & -i\lambda\frac{\sin\hat{\Phi}_n t}{\hat{\Phi}_n}\hat{A} \\ -i\lambda\frac{\sin\hat{\Phi}_{n-1}t}{\hat{\Phi}_{n-1}}\hat{A}^\dagger & \cos\hat{\Phi}_{n-1}t + i\frac{\hat{\Lambda}_{n-1}}{2\hat{\Phi}_{n-1}}\sin\hat{\Phi}_{n-1}t \end{pmatrix}. \tag{23}$$

In the time evolution operator, the generalized detuning Λ_n and generalized Rabi frequency Φ_n are:

$$\hat{\Lambda}_n = \omega - \omega_0[\hat{A},\hat{A}^\dagger], \qquad [\hat{A},\hat{A}^\dagger] = 1 + 2\frac{\chi}{\omega_0}\hat{n}, \tag{24}$$

$$\hat{\Phi}_n = \sqrt{\lambda^2\hat{A}\hat{A}^\dagger + \hat{\Lambda}_n^2/4} = \sqrt{\lambda^2(\hat{n}+1)f^2(\hat{n}+1) + \hat{\Lambda}_n^2/4}, \tag{25}$$

and $\omega_0[\hat{A},\hat{A}^\dagger] \equiv \omega_0(n)$ can be interpreted as an intensity dependent frequency. It is worth mentioning that the expression (23) is the general form of the evolution operator for a deformed-oscillator algebra and as a valuable consequence, setting $f(\hat{n}) = 1$ or equivalently in our work $\chi = 0$ or $k = 1$ in (23), (24) and (25) we can recover the standard JCM formalism at non exact resonance (Vidiella-Baranco et al. 1992; Dodonov and Man'ko 2003).

We recall that in the f-deformed JCM unlike the standard JCM there is a critical point for the generalized Rabi frequency Φ_n as a function of the photon number n. In fact Φ_n in Eq. (25) has the minimum value at special $n_{\min}$ (de los Santos-Sanches and Recamier 2012):

$$n_{\min} = \frac{1}{2}\frac{\chi\Delta - \left(1+\frac{\chi}{\omega_0}\right)\lambda^2}{\chi^2 + \frac{\chi}{\omega_0}\lambda^2}, \tag{26}$$

where $\Delta = \omega - \omega_0$ is the detuning between the atom and the field. It will be specified that the existence of such critical point in generalized Rabi frequency can yield some important differences between the nonlinear and linear systems specially from the point of view of the nonclassical properties. In fact, the unusual statistical behavior around $n_{\min}$, which explicitly depends on the χ and Δ, has been our main motivation to study the effects of the field intensity as well as detuning parameter in our model.

For simplicity we rewrite the time evolution operators in terms of new parameters as follow:

$$\hat{U}(t) = \begin{pmatrix} \hat{C}_{n+1}(t) & -i\lambda\hat{D}_{n+1}(t)\hat{A} \\ -i\lambda\hat{D}_n(t)\hat{A}^\dagger & \hat{C}_n^\dagger(t) \end{pmatrix}, \tag{27}$$

where

$$\hat{C}_{n+1}(t) = \cos\hat{\Phi}_n t - i\frac{\hat{\Lambda}_n}{2\hat{\Phi}_n}\sin\hat{\Phi}_n t, \quad \hat{D}_{n+1}(t) = \frac{\sin\hat{\Phi}_n t}{\hat{\Phi}_n}. \tag{28}$$

Now we can specify the time dependent density operator for the composed system. By using Eqs. (21), (22), (27) and (28) we will have:

$$\hat{\rho}(t) = \begin{pmatrix} \hat{\mathcal{R}}\hat{\rho}_F(0)\hat{\mathcal{R}}^\dagger & \hat{\mathcal{R}}\hat{\rho}_F(0)\hat{\mathcal{S}}^\dagger \\ \hat{\mathcal{S}}\hat{\rho}_F(0)\hat{\mathcal{R}}^\dagger & \hat{\mathcal{S}}\hat{\rho}_F(0)\hat{\mathcal{S}}^\dagger \end{pmatrix}, \tag{29}$$

where

$$\hat{\mathcal{R}} = \hat{C}_{n+1}(t), \quad \hat{\mathcal{S}} = -i\lambda\hat{A}^\dagger\hat{D}_{n+1}(t). \tag{30}$$

In this article, we assume that the initial single-mode of the electromagnetic field state inside the cavity is in the superposed state in (10) for which the density operator is:

$$\begin{aligned}\hat{\rho} = |N_s|^2\big(&|\alpha, f\rangle\langle\alpha, f| + e^{-i\phi}|\alpha, f\rangle\langle\alpha e^{i\theta}, f| \\ &+ e^{i\phi}|\alpha e^{i\theta}, f\rangle\langle\alpha, f| + |\alpha e^{i\theta}, f\rangle\langle\alpha e^{i\theta}, f|\big).\end{aligned} \tag{31}$$

Now replacing (31) as the initial cavity field density operator in (29) and using the relations (28) and (30) we can find the explicit form of the atom-field interaction system density operator $\hat{\rho}(t)$. Therefore the expectation values and the quantities which contain all the information about the under review system can be determined. For example one of the most important quantities in such a system of interacting atom and field is the field photon number distribution which at any arbitrary time t is defined as:

$$P_n(t) = \langle n|\hat{\rho}_F(t)|n\rangle, \tag{32}$$

where the density operator of the cavity field in time t can be easily obtained by the tracing the composed system density operator over the atomic variables in (29), as follows:

$$\hat{\rho}_F(t) = Tr_A[\hat{\rho}(t)] = \hat{\mathcal{R}}\hat{\rho}_F(0)\hat{\mathcal{R}}^\dagger + \hat{\mathcal{S}}\hat{\rho}_F(0)\hat{\mathcal{S}}^\dagger. \tag{33}$$

Regarding the dynamics of the system which is very susceptible to the photon statistics of the initial cavity field, the photon number distribution of the field at $t = 0$ [the probability of finding n photon in the state (10)] is defined as:

$$P_n(0) = \langle n|\hat{\rho}_F(0)|n\rangle = \frac{4|N_s|^2|N_f|^2|\alpha_0|^{2n}}{n!\,[f(n)f^\dagger(n)]!}\cos^2\left(\frac{\phi + n\theta}{2}\right). \tag{34}$$

Considering the condition $\chi/\omega_0 \ll 1$ in (16), one can easily find that $|\alpha_0|^2$ in above expression is approximately equal to the initial mean photon number of the cavity field; therefore, hereafter we would simply use $|\alpha_0|^2$ as a reference to the initial mean photon number of the cavity field. Also we use (31) as the initial field density operator $\hat{\rho}_F(0)$. It is easy to see that the above expression is exactly (12) in which $\alpha = \alpha_0$.

In our model, we consider the general form of the superposition of NCSs in (10) and (31) with arbitrary phase differences; therefore this model can be quite general. However, for our numerical analyses we confine ourselves to the special choices for ϕ and θ. These choices result in the particular category of superposition of NCSs. The choices are $\phi = 0$ and $\theta = \pi$ which gives us the even nonlinear coherent state (ENCS) for the initial cavity field, so our main problem is the interaction of a single two-level atom with an even nonlinear coherent state in a lossless optical cavity. It is evident that due to the ability of a nonlinear Kerr medium as a practical source to generate the superposition of CSs (NCSs) (Yurke and Stoler 1986), the f-deformed JCM with the nonlinearity function (16) can be considered as an experimental scheme for studying the interaction of a single two-level atom with an ENCS. By using the nonlinearity function (16) with $k = 2$ in (9) and (11) and setting $\phi = 0$ and $\theta = \pi$, the normalization constants in nonlinear system are given by:

$$\begin{aligned} N_f &= \sqrt{\frac{1 - (\chi/\omega_0)}{{}_0F_1(\omega_0/\chi; (\omega_0/\chi)|\alpha|^2)}} \\ N_s &= \sqrt{\frac{{}_0F_1(\omega_0/\chi; (\omega_0/\chi)|\alpha|^2)}{2\left({}_0F_1(\omega_0/\chi; (\omega_0/\chi)|\alpha|^2) + {}_0F_1(\omega_0/\chi; -(\omega_0/\chi)|\alpha|^2)\right)}}, \end{aligned} \tag{35}$$

where ${}_0F_1(a; x)$ is the confluent hypergeometric function. Now by setting (35) in (34) we can obtain the photon number distribution of the initial cavity field for the nonlinear case under consideration:

$$P_{2n}(0) = \frac{2(\omega_0/\chi)^n |\alpha_0|^{2n}}{n!\left({}_0F_1(\omega_0/\chi; (\omega_0/\chi)|\alpha|^2) + {}_0F_1(\omega_0/\chi; -(\omega_0/\chi)|\alpha|^2)\right)(\omega_0/\chi)_n} \tag{36}$$

$$P_{2n+1}(0) = 0,$$

where $(a)_n = \Gamma(a+n)/\Gamma(a)$ is the Pochhammer symbol. On the other hand for understanding the effect of nonlinearity on both the initial cavity field and the atom-field coupling, we need to compare the interaction of a single two-level atom with single-mode cavity field in presence of nonlinearity and without it. In fact, by setting $f(\hat{n}) = 1$ we would deal with the problem of a two-level atom interacting with cavity field initially in a general superposition of standard CSs with the form of (13). In the absence of nonlinearity, where there is not any nonlinear Kerr-like medium inside the cavity or any intensity-dependent coupling, it is clear that the dynamics of the system is described by the Hamiltonian (14). Also the state of (13) for the mentioned choices of the phase differences, $\phi = 0$ and $\theta = \pi$, is an even CSs (for future references denoted by ECS), so in the absence of nonlinearity and for these choices of ϕ and θ we deal with the problem of the interaction of a single two-level atom with an even coherent state which has been already studied for exact resonance in some works (Gerry and Hach 1993; Vidiella-Baranco et al. 1992). We recall that a convenient experimental scheme for this type of interaction can be found in Brune et al. (1992) where an immediate injection of atoms in resonance with the cavity field would interact with the superposition of CSs produced by the atoms very detuned from the cavity resonance frequency. In the linear system the normalization constants and initial photon number distribution of the cavity field are obtained by setting $f(n) = 1$ or equivalently $\chi = 0$ in (9), (11) and (34). Henceforth we choose $\hbar = 1$ and $\omega_0 = 1$ for simplicity.

3 Evolution of atomic inversion

The atomic inversion which is defined as the difference between the probabilities of the atom being in the excited state and the ground state, is one of the important quantities in the context of atom-field interactions. It is found that this quantity is very sensitive to the statistical properties of the initial radiation field. For instance if the field is initially prepared in the number state the atomic inversion shows the oscillations with the Rabi frequency like the semiclassical predictions. However, when the initial field is a coherent state, we will have collapse-revival phenomenon as a fully quantum mechanical feature which has been already experimentally verified Rempe et al. (1987). Studying collapse and revival times of oscillations in the atomic inversion is a convenient way to have a good understanding of the atom-field interaction system. Knowing that the atomic inversion is defined as the expectation value of the atomic transition operator $\hat{\sigma}_3$ and applying (29), we can write:

$$\begin{aligned}\langle\hat{\sigma}_3\rangle &= Tr[\hat{\rho}(t)\hat{\sigma}_3] \\ &= \sum_{n=0}^{\infty}\Big(\langle n|\hat{\mathcal{R}}\hat{\rho}_F(0)\hat{\mathcal{R}}^{\dagger}|n\rangle - \langle n|\hat{\mathcal{S}}\hat{\rho}_F(0)\hat{\mathcal{S}}^{\dagger}|n\rangle\Big).\end{aligned} \tag{37}$$

Inserting Eqs. (28) and (30) into Eq. (37), we have the explicit form of the atomic inversion as:

$$\langle\hat{\sigma}_3\rangle = \sum_{n=0}^{\infty} P_n(0)\left[\frac{{\Lambda_n}^2}{4\Phi_n^2} + \frac{\lambda^2(n+1)f^2(n+1)}{\Phi_n^2}\cos(2\Phi_n t)\right], \tag{38}$$

where $P_n(0)$ is the initial photon number distribution of the cavity field and it is given by (36). It should be noted that in our numerical investigations, we would plot the time evolution of the parameters as functions of time t instead of scaled time $T = \lambda t$. This is due to the fact that in f-deformed JCM, it is not possible to factorize the product λt and the overall dynamics of the system is not independent of the coupling constant λ. Due to the importance of the collapse and revival phenomena in all aspects of the contents of atom-field interaction, we can evaluate the approximate revival time which is the time associated with the revival of sinusoidal Rabi oscillations. Considering that the revivals occur when the phase difference of the oscillations of the neighbour terms in summation (38) is 2π, we have:

$$t_r \approx \frac{\pi}{\Phi_{\bar{n}+2} - \Phi_{\bar{n}}}, \tag{39}$$

where Φ_n is given by Eq. (25) and we put $n \approx \bar{n} \approx |\alpha_0|^2$. Also please note that for an ENCS as the initial cavity field, $P_n(0)$ in (38) is zero for odd number of photons "n", therefore the nonzero neighboring terms in (38) are Φ_{n+2} and Φ_n instead of Φ_{n+1} and Φ_n.

In Figs. 1 and 2, we have plotted the temporal evolution of the atomic inversion with the cavity field initially in an ENCS (ECS) in left (right) plots when $\Delta \neq 0$ corresponding to Fig. 1 and $\Delta = 0$ corresponding to Fig. 2. In addition to the study of nonlinearity effects by comparing linear and nonlinear systems in two resonant and nonresonant cases, the effect of increasing of the initial mean photon number $|\alpha_0|^2$ has also been investigated. Having looked at our curves, we can summarize the results as (1) a comparison between the linear and nonlinear systems shows that the structure of the oscillations in the nonlinear case (ENCS as the initial field), is much more complex than the linear one (ECS as the initial field), which is due to influence of anharmonicity parameter χ through the generalized Rabi frequency (2) in the presence of nonlinearity and when $\Delta \neq 0$ collapse-revivals phenomena show themselves at higher field intensities (Fig. 1a, c in comparison to Fig. 1e) and increasing the initial field photon number cause the variations to occur in the positive

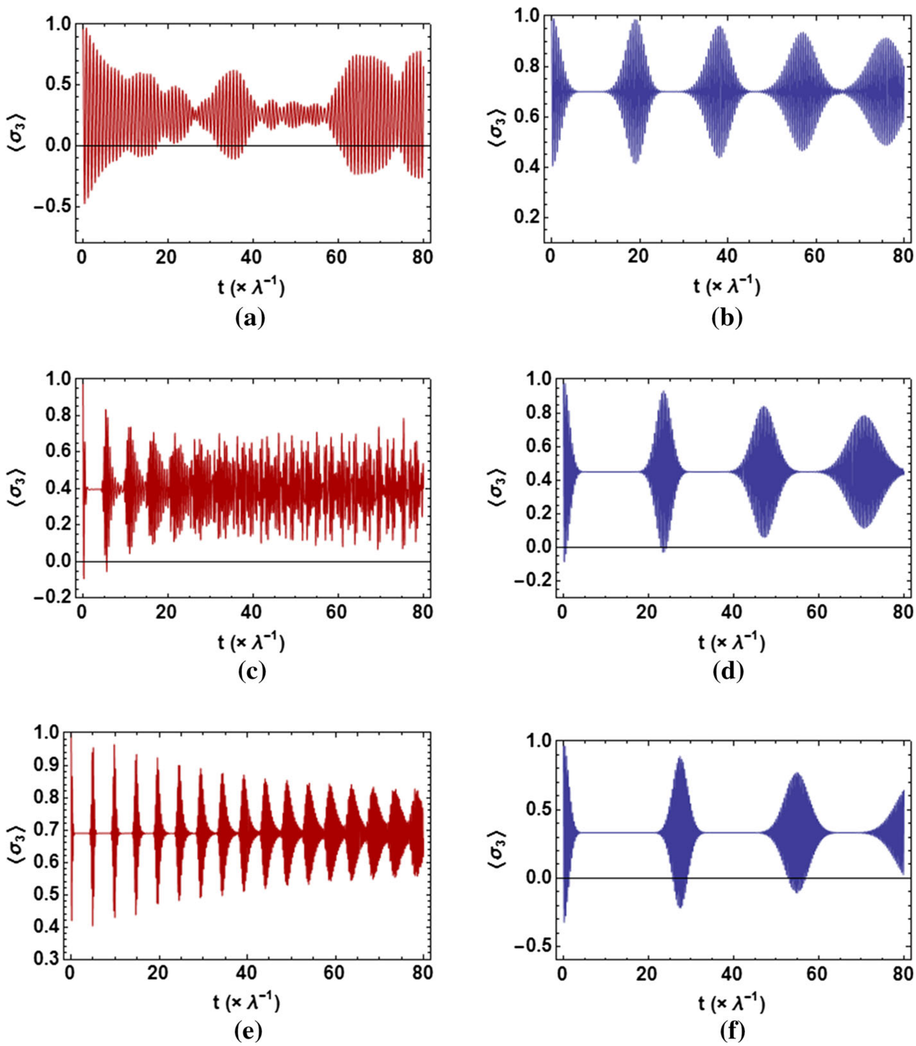


Fig. 1 The evolution of atomic inversion $\langle\hat{\sigma}_3\rangle$ against time t. The parameters are $\Delta = 0.03, \lambda = 0.003$ and the field initially prepared in an ENCS with (**a**) $\chi = \lambda/3, |\alpha_0|^2 = 10$ (**c**) $\chi = \lambda/3, |\alpha_0|^2 = 30$ and (**e**) $\chi = \lambda/3, |\alpha_0|^2 = 50$. Field initially prepared in an ECS with (**b**) $\chi = 0, |\alpha_0|^2 = 10$ (**d**) $\chi = 0, |\alpha_0|^2 = 30$ and (**f**) $\chi = 0, |\alpha_0|^2 = 50$

region which is opposite the linear case where by increasing the initial field intensity it is feasible for the oscillations to be in the negative region (compare the left and right columns of Fig. 1), (3) in linear and nonlinear systems, for both resonant and nonresonant cases, increasing the initial mean photon number results in an increase in the collapses times; this indicates that the mean number of isolated revivals is decreased. This phenomena is in agreement with the results of Eberly et al. (1980) in which the authors showed that only for large enough values of the mean photon number (as well as large enough values of the

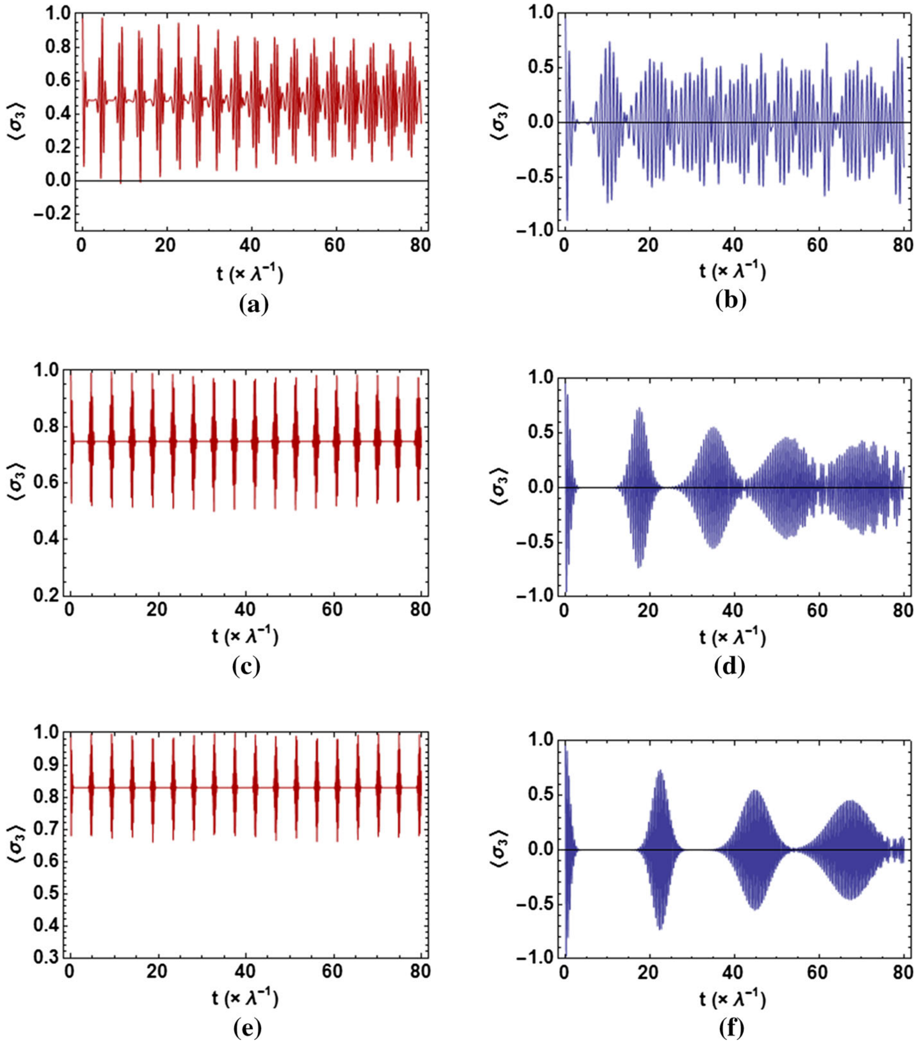


Fig. 2 The same as Fig. 1 except that $\Delta = 0$

scaled time) the collapses and revivals become recognizable, (4) the influence of detuning in linear system is twofold: The presence of Δ essentially increases the time interval of the consecutive revivals and collapses Eberly et al. (1980). Secondly, in non exact resonance, the probability for transferring the population between the exited and the ground states of the atom becomes smaller. This results in the population inversion to have a non-zero average instead of zero which is the average of $\langle \hat{\sigma}_3 \rangle$ when $\Delta = 0$. The dependence of the generalized Rabi frequency in f-deformed JCM on the anharmonicity parameter χ through the nonlinearity function (16) causes different behavior of the atomic inversion in non-linear system in comparison to the linear one. It is clear from the left plots of Fig. 2 that even when $\Delta = 0$, the variation of $\langle \sigma_3 \rangle$ is completely in positive region for all field intensities. Also, in contrast with the linear system, increasing the initial field intensity in the nonlinear system would reduce the domain of oscillations (compare the left and right columns of Fig. 2).

4 Field dynamics

4.1 Statistical properties of the field

In order to attain more information regarding the statistical properties of the field in the interaction time, we use the so-called Mandel q-parameter Mandel (1979) which is the normalized variance of the photon distribution and is given by:

$$q = \frac{\langle (\Delta\hat{n})^2 \rangle - \langle \hat{n} \rangle}{\langle \hat{n} \rangle}, \tag{40}$$

where $(\Delta\hat{n})^2 = \langle \hat{n}^2 \rangle - \langle \hat{n} \rangle^2$. It should be noted that depending on the value of Mandel parameter, the field may exhibit Poissonian, super-Poissonian or sub-Poissonian statistics for $q = 0, q > 0$ and $q < 0$ respectively. In (40) we can calculate the expectation value of any field operator $\hat{O}$ by:

$$\langle \hat{O} \rangle = Tr[\hat{\rho}(t)\hat{O}], \tag{41}$$

where $\hat{\rho}(t)$ is given by (29). So the expectation values we need, can be easily obtained as:

$$\langle \hat{n} \rangle = \sum_{n=0}^{\infty} P_n(0) \left[n + \left(1 - \frac{\Lambda_n^2}{4\Phi_n^2} \right) \sin^2(\Phi_n t) \right] \tag{42}$$

$$\langle \hat{n}^2 \rangle = \sum_{n=0}^{\infty} Pn(0) \left[n^2 + (2n+1) \left(1 - \frac{\Lambda_n^2}{4\Phi_n^2} \right) \sin^2(\Phi_n t) \right] \tag{43}$$

In Figs. 3 and 4, the temporal evolution of the Mandel parameter is plotted against time t for which the cavity field is initially prepared in an ENCS (ECS) corresponding to the left (right) plots and the detuning parameter has been chosen to be nonzero in Fig. 3 and zero in Fig. 4. In Fig. 3a when the initial mean photon number is $|\alpha_0|^2 = 10$, the temporal evolution of the q-parameter of the nonlinear system oscillates between positive and negative values. However, in the linear case the field almost always shows a super-Poissonian statistics (see Fig. 3b). But by increasing the initial field intensity the behavior of the nonlinear system is completely different from the linear one. It is seen from Fig. 3c, e that in the interaction of a single two-level atom with an ENCS (nonlinear system), increasing the field intensity makes all the q-parameter oscillations take negative values; this means that at all interaction time the field indicates a sub-Poissonian statistics which is obviously an important nonclassical feature while in linear system for the same values of the field intensities, the statistical behavior of the field tend to vary around a Poissonian statistics (for instance see Fig. 3f). It should be mentioned that in the nonlinear case, we have chosen the parameters χ, Δ and λ in a way that $n_{\min}$ in (26) be around 10. Therefore the different behaviors of the statistical features of the cavity field in the nonlinear system (described by f-deformed JCM) can be understood from the point that unlike the standard JCM (linear system), there exists an unusual statistical behavior above the $n_{\min}$ depending

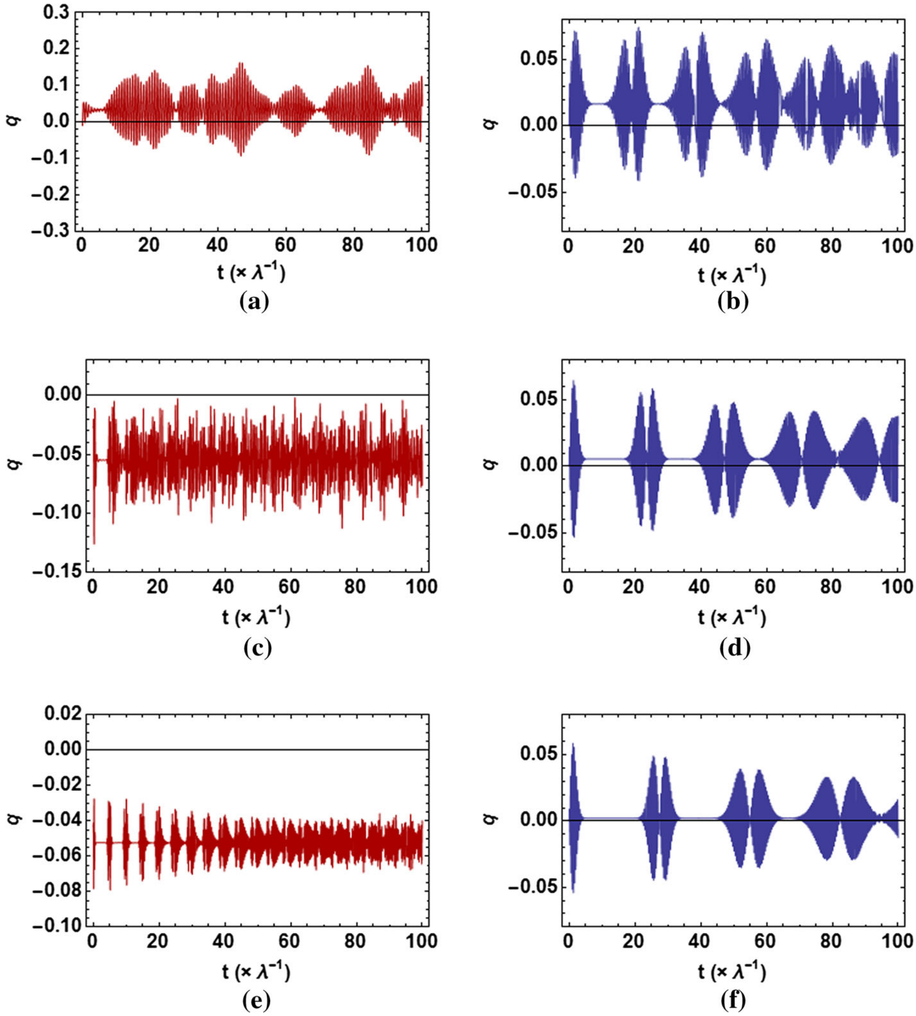


Fig. 3 The same as Fig. 1 but for the time evolution of Mandel q-parameter

on the initial field mean photon number. Also please note that the collapse-revival phenomena become more and more apparent in the nonlinear system as the field intensity increases (see the left column of Fig. 3). In fact due to the nonlinearity nature of the system under consideration, an increase in the field intensity lead to changing the chaotic behaviour of Mandel q-parameter around zero value (Poissonian statistics) to nearly regular oscillations in the negative region (sub-Poissonian statistics). On the other hand the comparison between Figs. 3 and 4 can indicates the effect of detuning on the interaction systems. Looking at the left column of Fig. 4 shows that when the cavity field and the atom are in exact resonance, for all considering initial field intensities the Mandel q-parameter takes negative values at almost all intervals of time which indicates the nonclassicality feature of the field during the interaction time. The fact that in the exact resonance, the field of the nonlinear system exhibits sub-Poissonian statistics for all interaction times even in low intensity (which do not happen when $\Delta \neq 0$), can be explained in this way that when

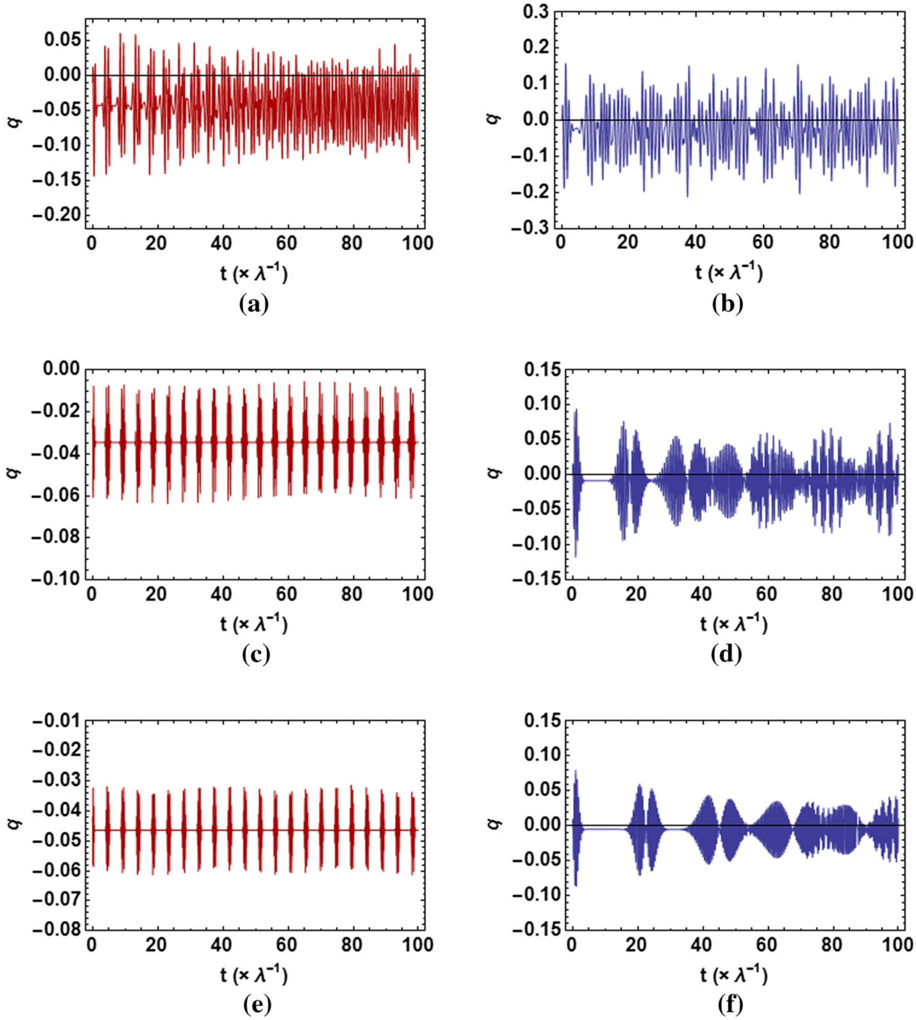


Fig. 4 The same as Fig. 3 except that $\Delta = 0$

$\Delta = 0$, there is no meaningful value for critical photon number $n_{\min}$ which the peculiar statistical behaviour occur above it (compare Figs. 3a, 4a). Also it is seen from the left-hand side plots of Fig. 4 that when $\Delta = 0$ the increase of the field intensity lead to decreases the maximum values of the Mandel parameter (increase the negativity) which is equivalent to the strength of nonclassicality of the field under the interaction. Thus in general we may conclude that the lack of detuning as well as nonlinearity nature of the system has an important role in revealing this nonclassical property. The right plots of Figs. 3 and 4 show that the linear system in exact resonance indicates nearly the same behavior with the system in nonresonant case i.e. the Mandel parameter of the field oscillates between positive and negative values. The only difference is that the mean value of the variations (which the Mandel parameter oscillates around it) is a very small negative amount. In addition in resonant case the typical fractional collapses and revivals will appear in high intensities (compare Fig. 4b with Fig. 4d, f). Therefore one can be certain that in the interaction of a single two-level atom with an ECS (against the nonlinear system), the detuning parameter as well as changing of the initial field intensity do not have a significant influence on the Mandel parameter evolution or equivalently the nonclassicality of light.

4.2 Entropy evolution of the field

In recent years, It is widely accepted that quantum entanglement is one of the striking feature of quantum optics and especially in quantum information science (Bennett 1995; Bennett and Divincenzo 1995; Bennett and Divincenzo 2000; Naderali et al. 2013). The quantum dynamics described by the atom-field interaction in a cavity QED is a simple way to produce the quantum entangled state which can be considered as an essential component for implementation of quantum information processing devices (Werner 2001). On the other hand the time evolution of the entropy of the field or atom is a standard way to imply the strength of entanglement (Mortezapour et al. 2015). In fact the higher (lower) the entropy the greater (smaller) the degree of entanglement. According to von Neumann entropy, the entropies of the atom and the field as separated system, are defined by:

$$S_{A(F)}(t) = -Tr_{A(F)}(\hat{\rho}_{A(F)}(t) \ln \hat{\rho}_{A(E)}(t)) \tag{44}$$

Having $\hat{\rho}^n = \hat{\rho}$ (where n is an integer), or equivalently $Tr\hat{\rho}^2 = 1$, for a pure state, one can easily see that entropy should be zero for such states. On the other hand, for a mixed state, we have $Tr\hat{\rho}^2 < 1$ and the entropy would no longer be zero. It is worth mentioning that according to Araki and Lieb theorem (Araki and Lieb 1970) for a two components quantum systems for example our review atom-field interaction system, the entropies are limited by the triangle inequality as follow:

$$|S_A(t) - S_F(t)| \leq S(t) \leq |S_A(t) + S_F(t)|, \tag{45}$$

where we note the total entropy of the composite atom-field system at any time t by $S(t)$. As a result if the system is initially prepared in a pure state (as it is supposed in our system under consideration) then at any time $t > 0$ the reduced entropies of the field and the atomic subsystems are identical which means $S_A(t) = S_F(t)$. Therefore we can focus on the evolution of the field entropy instead of atomic entropy.

Linear entropy is the first order approximation of the entropy; and, in our work, this approximation would be a very good substitute for the entropy. The linear entropy would be defined as:

$$\xi = 1 - Tr_F[\hat{\rho}_F^2(t)] \tag{46}$$

Also taking advantage of linear entropy instead of entropy, we could perform comparisons between our results and the results of Vidiella-Baranco et al. (1992).

It is obvious that the amount of linear entropy can serve as a measure of the field purity (or equivalently measure of the entanglement). In fact we have $\xi = 0$ for a pure state and $\xi > 0$ (which happens for example in a statistical mixture) for less pure states. The quantity of $Tr_F[\hat{\rho}_F{}^2(t)]$ is easily obtained from Eq. (33) and for the field initially prepared in an ENCS, it will be as the following form:

$$\xi = 1 - 4|N_s|^4(\zeta_1{}^2 + \zeta_2{}^2 + 2|\zeta_3|^2), \tag{47}$$

where by using (30), (31) and (33) we have:

$$\zeta_1 = \sum_{n=0}^{\infty} p_n(\eta_n \, |C_{n+1}|^2)$$
$$\zeta_2 = \sum_{n=0}^{\infty} p_n\left(\kappa_n \, \frac{\lambda^2 \, (nf^2(n))^2 \, |D_n|^2}{\alpha^2}\right) \tag{48}$$
$$\zeta_3 = \sum_{n=0}^{\infty} p_n(\eta_n \, C_{n+1})\left(\kappa_n \, \frac{i\,\lambda\, nf^2(n)\, D_n}{\alpha}\right),$$

with $\eta_n = [1 + (-1)^n], \kappa_n = [1 - (-1)^n]$ and p_n is the probability of finding n photon in a nonlinear coherent state in (8) which is given by:

$$p_n = |\langle n|\alpha, f\rangle|^2 = \frac{|N_f|^2 |\alpha|^{2n}}{n! \, [f^2(n)]!} \tag{49}$$

The numerical results of the temporal evolution of the linear entropy are illustrated in Figs. 5 and 6 where, in the left plots, the initial state of the cavity field is an ENCS with $\lambda = 0.003, \chi = \lambda/3$ and in the right plots we investigate the linear system in which the initial state of the cavity field is considered to be an ECS with the same amount for λ but $\chi = 0$. Also we consider the effect of detuning in Fig. 5 by setting $\Delta = 0.03$ while the linear entropy of the field in exact resonance with the atom ($\Delta = 0$) has been studied in Fig. 6. It is clear that in nonlinear system and in the nonresonant condition a chaotic behavior in the evolution of the field entropy can be observed for lower intensities whereas by increasing of the field intensity the collapse and revival patterns are revealed (see the left column in Fig. 5). Furthermore increasing the field intensity reduces the maximum value of the quantum field entropy (compare Fig. 5e with Fig. 5a, c). In the exact resonant case ($\Delta = 0$) the behavior of the field entropy evolution is different and a regular oscillatory behavior is observed especially in higher intensities (left column in Fig. 6). We notice also in this case increasing the field intensity lead to decrease of the quantum field entropy which means the degree of entanglement between the atom and the field reduces (Fig. 6c, e). We note that the zero value in these plots indicates that the atom and the field are disentangled. In fact at times when the entropy attains the zero value, the atom is in its upper or lower states (a pure state). It should be noted that there is a connection between the field purity during the interaction and its quantum statistical properties Moya-Cessa and Vidiella-Baranco (1992). In fact, when the field indicates sub-Poissonian characteristics, both the atom and the field could reversibly get back to the initial pure states at specific time intervals. The behavior of the field entropy evolution in the nonlinear system can be explained by comparing the left plots of Figs. 5 and 6 with the respective column in Figs. 3 and 4. It is clear from Fig. 5a that it is difficult for the system to reconstruct a pure state because the field is nearly super-Poissonian (see Fig. 3a). By increasing the field intensity, it is possible for the field to be near a pure state approximately at the corresponding revival times of atomic inversion because the field is extremely sub-Poissonian (see Fig. 3c, e). This situation is more apparent for $\Delta = 0$ when for all the considered intensities, the field entropy has strong oscillations whose minimum values are located at the corresponding revival times for which the cavity field shows sub-Poissonian photon statistics (see the left plots of Figs. 4, 6).

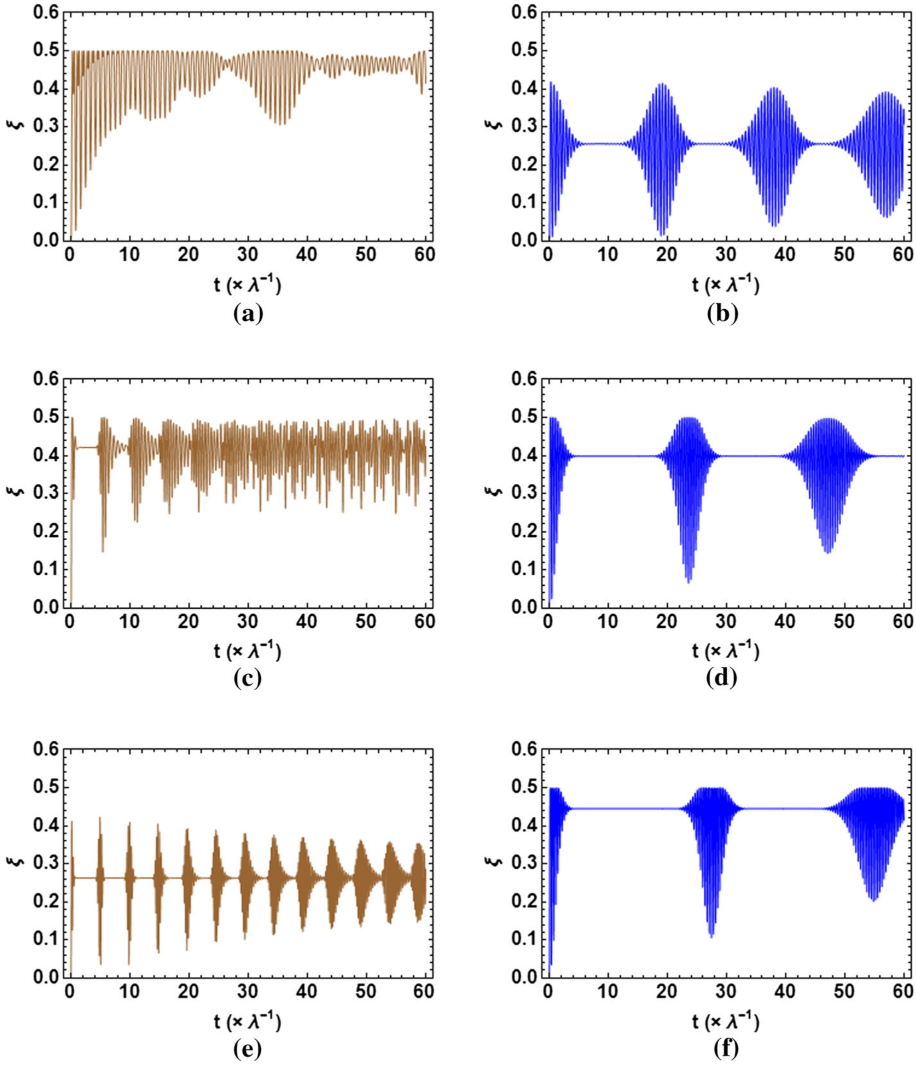


Fig. 5 The same as Fig. 1 but for the time evolution of field entropy

It also should be mentioned that for lower intensities in the left column of Fig. 5, there is a maximum line around $\xi = \frac{1}{2}$ for field entropy which is the value of ξ for the field initially prepared in a statistical mixture of NCSs with the form of:

$$\rho = \frac{1}{2}|\alpha, f\rangle\langle\alpha, f| + \frac{1}{2}| - \alpha, f\rangle\langle-\alpha, f|, \tag{50}$$

which can be explained in this way that the ENCS as a superposition of NCSs decays to a statistical mixture with minimum value of purity during the interaction. In linear case when cavity field is prepared in an ECS the temporal evolution of the field entropy shows completely different behavior. It can be seen from the right plots of Fig. 5 that in presence of detuning, the entropy of the field shows a kind of oscillations (collapse and revival

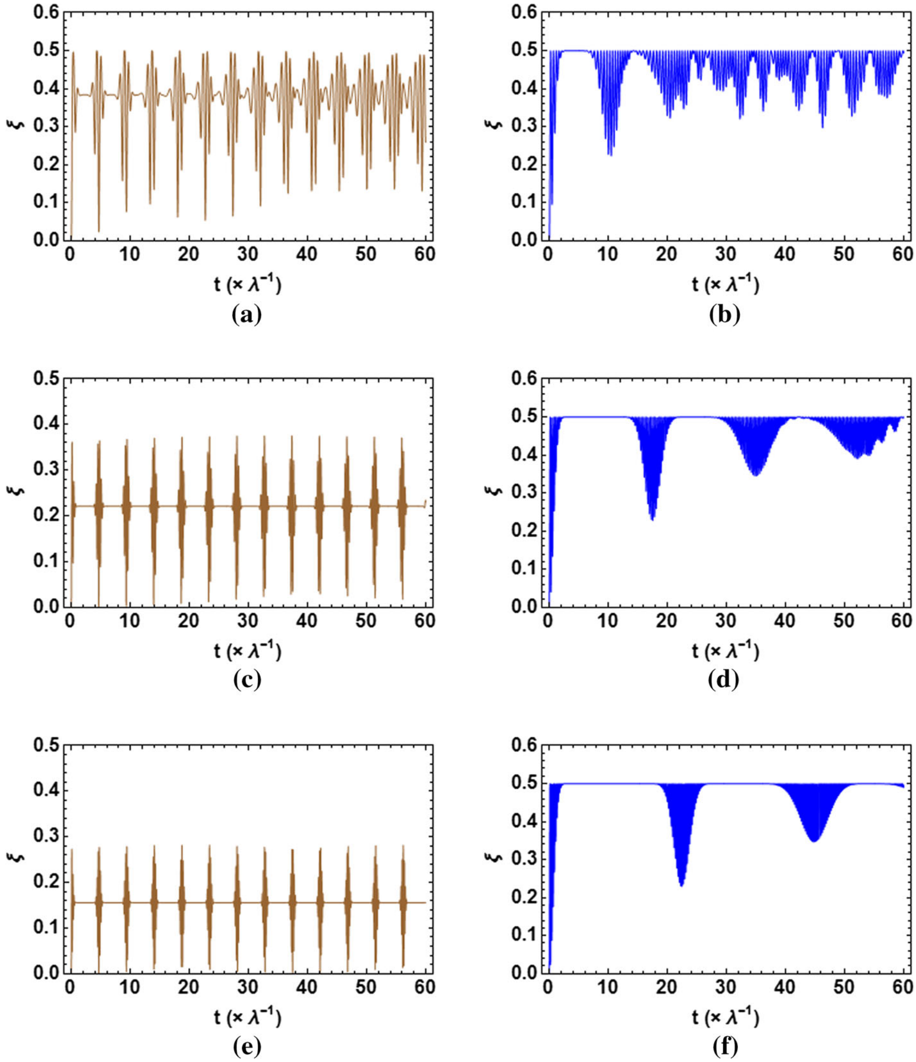


Fig. 6 The same as Fig. 5 except that $\Delta = 0$

patterns) while a chaotic behavior is revealed in exact resonant case (right column in Fig. 6). Furthermore against nonlinear case when the field intensity increases in nonresonant case, the quantum field entropy or equivalently the degree of entanglement between the atom and the field has been increased (compare the right plots of Fig. 5 with the left one). Also in exact resonance the change of field intensity do not change the maximum value of quantum entropy of the field Vidiella-Baranco et al. (1992) and there is a very well line around the $\xi = \frac{1}{2}$ which is related to a statistical mixture of CSs [setting $f(n) = 1$ in (50)].

Fig. 7 Evolution of the Q-function for cavity field when initial field is an ENCS, $\Delta = 0.03$ in *left column*, $\Delta = 0$ in *right column* and $t = 0$ for (**a**) and (**b**), $t = t_r/2$ for (**c**) and (**d**), $t = t_r$ for (**e**) and (**f**) and $t = 3t_r/2$ for (**g**) and (**h**). The horizontal axis is Re(β) and the vertical axis is Im(β) and the other parameters are $\chi = \lambda/3, \lambda = 0.003, |\alpha_0|^2 = 50$

4.3 Evolution of Q-function

The time-dependent quasi-probability distributions can be useful in the describing the electromagnetic field modes and hence obtaining more worthwhile information about the atom-field interaction system (Scully and Zubairy 2001). We could analyze the field dynamics by studying the quantum state of the cavity field. In order to do so, we consider the evolution of the Q-function in phase-space which is defined by:

$$Q(\beta) = \frac{1}{\pi}\langle\beta|\hat{\rho}_F(t)|\beta\rangle, \tag{51}$$

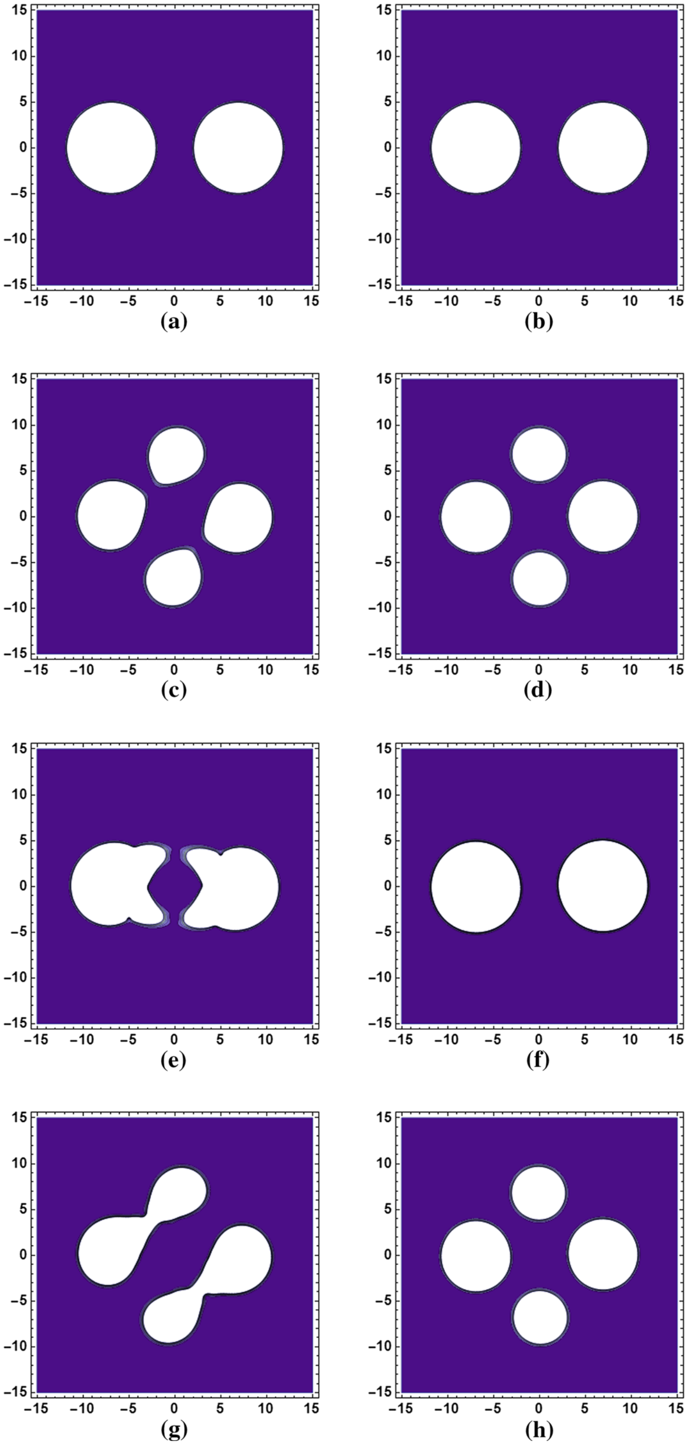
where as mentioned before, $\hat{\rho}_F(t)$ is the reduced density operator of the field given by (33) and the average is taken between the standard coherent state $|\beta\rangle$. By considering the relation (33) with (30) and (31), the explicit form of the Q-function is obtained as:

$$\begin{aligned} Q(\beta) = \frac{|N_s|^2}{\pi}(|\mathcal{Q}_1|^2 + |\mathcal{Q}_2|^2 + |\mathcal{Q}_3|^2 + |\mathcal{Q}_4|^2 \\ + \mathcal{Q}_1\mathcal{Q}_2^* + \mathcal{Q}_2\mathcal{Q}_1^* + \mathcal{Q}_3\mathcal{Q}_4^* + \mathcal{Q}_4\mathcal{Q}_3^*), \end{aligned} \tag{52}$$

where:

$$\begin{aligned} \mathcal{Q}_1 &= N_\alpha \sum_{n=0}^{\infty} \frac{\exp(-|\beta|^2/2)(\beta^*\alpha)^n}{n!\,[f(n)]!}\; C_{n+1}, \\ \mathcal{Q}_2 &= N_\alpha \sum_{n=0}^{\infty} \frac{\exp(-|\beta|^2/2)(\beta^*\alpha)^n(-1)^n}{n!\,[f(n)]!}\; C_{n+1}, \\ \mathcal{Q}_3 &= N_\alpha \sum_{n=0}^{\infty} \frac{\exp(-|\beta|^2/2)(\beta^*\alpha)^n f(n+1)}{n!\,[f(n)]!}\; (-i\lambda\beta^* D_{n+1}), \\ \mathcal{Q}_4 &= N_\alpha \sum_{n=0}^{\infty} \frac{\exp(-|\beta|^2/2)(\beta^*\alpha)^n f(n+1)\,(-1)^n}{n!\,[f(n)]!}\; (-i\lambda\beta^* D_{n+1}). \end{aligned} \tag{53}$$

and N_α is the normalization factor of a NCS which is given by (9). The contour plots of the cavity field Q-function are illustrated in Fig. 7 where we study the evolution of an ENCS in the interaction with the atom. The parameters are $\lambda = 0.003, \chi = \lambda/3$ and the initial mean photon number fixed at $|\alpha_0|^2 = 50$. In the left plots nonresonant case is considered with $\Delta = 0.03$ while the right plots concern the absence of the detuning parameter i.e. $\Delta = 0$. The parameters have been chosen in such a way that the revival times in (39) be around $t_r \approx 4.86/\lambda$ and $t_r \approx 4.68/\lambda$ respectively for nonresonant and resonant cases. As we expected if the field is initially at an ENCS, clearly we have two peaks at $t = 0$ corresponding to each constituent state (see Fig. 7a). It is clear in Fig. 7c that at $t_r/2$ i.e. at the first collapse time of atomic inversion (see Fig. 1e), the two well-localized starting peaks break into four components in the phase space which do not retain their original Gaussian form. In fact this phenomenon is related to the collapse of Rabi oscillations. At $t = t_r$ and $t = 3t_r/2$ for which there are respectively a revival and collapse in the atomic inversion, the components of the field state recombine to the two peaks (see Fig. 7e) and again split


(a)
(b)
(c)
(d)
(e)
(f)
(g)
(h)

into four not exactly separated components (see Fig. 7g). This means that we deal with a breaking of the initial two components of the field at $t = 0$ into more separated peaks on the time scale of the first collapse and then a recombination and a splitting of them for consecutive revivals and collapses. Indeed when we start with an ENCS, we have an f-deformed Schrödinger cat state with exactly two components and the same probabilities of finding the field in each of them as the initial cavity field. In Fig. 7c, g there exist multiple f-deformed Schrödinger cat-like states in later collapse times of the atomic inversion with four components due to the interaction with atom in presence of the Kerr-like nonlinearity. The separated components do not hold the same probabilities which is because of the intensity-dependent coupling between the atom and the field.

As a remarkable case it is of interest to consider the field state at the consecutive collapses of the atomic inversion when the atom and the cavity field is in exact resonance i.e. $\Delta = 0$. It is obvious from Fig. 7d, h that there exists a four-peak structure of the Q-function at $t = t_r/2$ and $t = 3t_r/2$ (at the collapse intervals of interaction time). Although we found that the components centred at $x = 0$ do not have the same probability with those centred at $y = 0$, but it is evident that these four well-localized Gaussian peaks correspond to the interference of the four-component NCSs $\pi/2$ out of phase with respect to each other. In fact in the interaction of a single two-level atom with an ENCS in the full nonlinear regime and the exact resonance condition we start with a f-deformed Schrödinger cat state which result from the interference of the states $|\alpha, f\rangle$ and $|-\alpha, f\rangle$. Then because of the interaction between the atom and the field we deal with four-component NCSs at the collapse times which arising from the interferences of $|\pm\alpha, f\rangle$ and $|\pm i\alpha, f\rangle$. Clearly such a superposition of NCSs can be interpreted as the deformed Schrödinger cat-like states but with superposition of four macroscopic states instead of two. It is worthwhile to mentioned that the Schrödinger cat states with two or more component are recognized as a resource for numerous quantum information processing tasks which are being considered in a lot of research both theoretically and experimentally. For instant the authors in Velastakis et al. (2013) created cat states as large as more than 100 photons and extended their protocol to create superpositions of up to four CSs (four-component Schrödinger cat states). This control creates a powerful interface between discrete and continuous variable quantum computation and could enable applications in metrology as well as quantum information processing. On the other side it has been found that the new concept of "four-photon nonlinear coherent states" which has been lately introduced by us as the eigenstates of the fourth-power of the generalized annihilation operator $\hat{A}^4$, can be obtained from the unique superpositions of the NCSs $|\pm\alpha, f\rangle$ and $|\pm i\alpha, f\rangle$ (Abbasi and Jafari 2016). In addition to the strongly oscillating photon count probability, it was shown that such states can exhibit some important nonclassical properties like sub-Poissonian photon statistics, the negativity of Wigner distribution function, amplitude-squared squeezing as well as higher-order squeezing. Considering that the four-photon NCSs can be arisen from the superposition of NCSs with the exact $\pi/2$ phase difference and on the other side the similarity of the Q-function of this superposition to the four-peak structure of the field in our system, which is evident in Fig. 7d, h, one can conclude that in exact resonance condition and at the collapse times of atomic inversion the under consideration system can be a possible source for such states. Indeed when the exact resonance condition is true, the initial cavity field is the eigenstate of the square of the generalized annihilation operator $\hat{A}^2$ while it can be the eigenstate of $\hat{A}^4$ at the first collapse time. The recombination and the splitting of the field components for successive revivals and collapses are clear in the right column of Fig. 7. Finally the same behavior comes along for the linear system as well, specially when $\Delta = 0$

the four-peak structure of the Q-function is very similar to that observed in a four-photon standard coherent state i.e. an eigenstate of $\hat{a}^4$ (Gerry and Hach 1993; Vidiella-Baranco et al. 1992).

5 Summary and conclusion

Developing a model which can be used to extract non-classical features of interacting atoms and fields is one of the most promising theoretical developments in Quantum Optics. In this paper, we investigate the interaction of a single two-level atom with a superposition of NCSs in the presence of additional Kerr-like medium and intensity dependent coupling. For this purpose we first introduce a general superposition of NCSs with two arbitrary phase factors in (10). The outstanding property of this superposition is that we are able to recover all types of superpositions of linear and nonlinear CSs with appropriate choices of relative phase factor and phase difference of amplitudes of the original components which are denoted by ϕ and θ respectively as well as nonlinearity function $f(n)$. After developing such a superposition method, we propose a solution for the f-deformed Jaynes–Cummings model based on the density operator method. It is worth mentioning that the real power of this method lies in the fact that it can be easily used for any mixture of field and atom initial states. Choosing the specific form of $f(n)$ in (16) with the special value of $k = 2$, guarantees specific forms of nonlinearity for both the field and intensity-dependent coupling between the atom and field. This means that firstly the Hamiltonian of the field becomes the Kerr Hamiltonian so in our system the atom is supposed to be surrounded by a Kerr-like medium. Secondly the coupling of the atom-cavity field is not constant and so we deal with an intensity-dependent atom-field coupling whose intensity dependence is determined by the function $f(n)$. Thirdly it can be possible to consider the cavity field as a nonlinear coherent state or a superposition of the NCSs because the medium is modeled by the deformed oscillator. On the other side using specific form of nonlinearity function (16) for describing the CSs of Morse-like potential make it possible to interpret the f-deformed JCM as a real quantum system involving the interaction of a single two-level atom with the Morse potential in which the quantized electromagnetic field can be considered as the NCSs or their superpositions. In order to study the influence of nonlinearity, the special features of our nonlinear system have been compared with the corresponding linear system. The dynamics of our nonlinear system is determined by the f-deformed JCM Hamiltonian (15) where the cavity field is initially prepared as a superposition of NCSs. On the other hand the corresponding linear system is easily recovered by substituting $f(\hat{n}) = 1$. Although this approach is quite general, for numerical analyses we considered the well known category of superposed states i.e. even NCSs (CSs). In addition to detuning parameter, the existence of a critical point (as a function of the photon number) in the generalized Rabi frequency of the f-deformed JCM, was the motivation to study the effect of the field intensity on the system quantum features. It is shown that the atomic inversion behavior in the interaction of the atom with the even NCSs is different from those which happens in linear system. For example increasing of the initial field intensity increases (decreases) the mean number of revivals in the nonlinear (linear) system. In addition it is found that in nonresonant case, for all the revivals variations to happen in the positive region, the nonlinear (linear) system must be in higher (lower) field intensities; however with the decrease (increase) of the initial field intensity, it is possible for the oscillations to happen in the negative region. As an important consequence, studying the statistical properties of the field via the well-known Mandel q-parameter showed that in both resonant and nonresonant cases our superposition of NCSs indicates nonclassical properties in its interaction with the atom while the superposition of the linear CSs did not indicate such properties in its interaction with the atom. In fact it was found that our nonlinear system can be considered as a source for generating the sub-Poissonian light especially when the cavity field and the atom are in exact resonance. Quantum entanglement between the atom and the field which is one of the main characteristics of quantum mechanics is studied in this paper via the time evolution of field entropy. Considering the effect of field intensity and detuning it is shown that the linear and nonlinear systems have completely opposite behavior. Increasing of the field intensity in nonlinear (linear) system lead to decrease (increase) of the field entropy or equivalently the degree of entanglement between the atom and the field . Also in the presence of detuning, the behavior of the field entropy is found to be chaotic (oscillatory) in the nonlinear (linear) system, while in exact resonant case the time evolution of nonlinear (linear) field entropy has oscillatory (chaotic) behavior during the interaction time. We found that, in our proposed system, there exists an interesting connection between the field purity and its sub-Poissonian characteristics. Finally the study of the Q-function dynamics for the deformed field showed the well-known splitting and then recombination of the field components in phase space which are related to the collapses and revivals of atomic inversion. We showed that multiple f-deformed Schrödinger cat states are developed as a result of the field evolution. The difference between the splitting phenomena related to the two resonant and nonresonant cases is that in resonant case (unlike the nonresonant case), the components of the field in phase space remain well-localized round hills with a Gaussian form at all times considered. Especially when $\varDelta = 0$ the initial two peaks in the phase space split to the four peaks at the collapses of atomic inversion, correspond to the four NCSs $\pi/2$ out of phase which can be interpreted as the f-deformed four-component Schrödinger cat-like states. Moreover the similarity of the Q-function structure of the field in our system and the “four-photon nonlinear coherent states”, reinforces this idea that the interaction of an ENCS with a single two-level atom inside a lossless optical cavity, provided in our model, may be the possible source of such states. Certainly the proof of this theory needs more accurate and detailed studies. Changing the phase parameters of the superposed states might affect the nonclassical characteristics of the interactions. Studying such effects could be the subject of further investigations. Since we have developed an interaction model based on the density operator, we can easily change the field initial state to any statistical mixture of NCSs and see how different choices change the interaction results. Moreover, the effects of Kerr medium as well as intensity-dependent coupling on the dynamics of an ENCS interacting with a single two-level atom can be separately investigated through the Hamiltonian (20). We would follow this path as a part of our future works. Finally it is worth mentioning that by setting $f(\hat{n}) = 1$ and $\varDelta = 0$, our presented formalism readily recovers the results of Gerry and Hach (1993), Vidiella-Baranco et al. (1992) as special cases.

Acknowledgments The authors would like to thank Prof M K Tavassoly from Yazd University for useful discussions. The comments of the respected reviewers were truly helpful in clarifying crucial parts of the research as well as in revealing novel and interesting persuits in future.

References

Abbasi, O., Jafari, A.: Four-photon nonlinear coherent states. J. Mod. Opt. (2016). doi:10.1080/09500340.2016.1206216

Abbasi, O., Tavassoly, M.K.: Superposition of two nonlinear coherent states $\frac{\pi}{2}$ out of phase and their nonclassical properties. Opt. Commun. **282**, 3737–3745 (2009)
Abbasi, O., Tavassoly, M.K.: Superpositions of the dual family of nonlinear coherent states and their non-classical properties. Opt. Commun. **283**, 2566–2574 (2010)
Abdel-Khalek, S., Berrada, K., Eleuch, H., Abel-Aty, M.: Dynamics of Wehrl entropy of a degenerate two-photon process with a nonlinear medium. Opt. Quantum Electron. **42**, 887–897 (2011)
Abdel-Khalek, S., Berrada, K., Obada, A.-S.F., Wahiddin, M.R.B.: Entropy squeezing for qubitfield system in the presence multi-photon process under decoherence effect. Opt. Quantum Electron. **47**, 267–278 (2015)
Agarwal, G.S., Puri, R.R.: Collapse and revival phenomenon in the evolution of a resonant field in a Kerr-like medium. Phys. Rev. A **39**, 2969–2977 (1989)
Araki, H., Lieb, E.: Entropy inequalities. Commun. Math. Phys. **18**, 160–170 (1970)
Bennett, C.H.: Quantum information and computation. Phys. Today **48**, 24–31 (1995)
Bennett, C.H., Divincenzo, D.P.: Towards an engineering era? Nature **377**, 389–390 (1995)
Bennett, C.H., Divincenzo, D.P.: Quantum information and computation. Nature **404**, 247–255 (2000)
Brune, M., Haroche, S., Raimond, J.M., Davidovich, L., Zagury, N.: Manipulation of photons in a cavity by dispersive atom-field coupling: quantum-nondemolition measurements and generation of "Schrödinger cat" states. Phys. Rev. A **45**, 5193–5214 (1992)
Buck, B., Sukumar, C.V.: Exactly soluble model of atom–phonon coupling showing periodic decay and revival. Phys. Lett. A **81**, 132–135 (1981)
Buzek, V.: Jaynes–Cummings model with intensity-dependent coupling interacting with Holstein–Primakoff SU(1,1) coherent state. Phys. Rev. A **39**, 3196–3199 (1989)
Crnugelj, J., Martinis, M., Mikuta-Martinis, V.: Jaynes–Cummings model and the deformed-oscillator algebra. Phys. Lett. A **188**, 347–354 (1994)
de los Santos-Sanches, O., Recamier, J.: The f-deformed Jaynes–Cummings model and its nonlinear coherent states. J. Phys. B: At. Mol. Opt. Phys. (2012). doi:10.1088/0953-4075/45/1/015502
de Matos Filho, R.L., Vogel, W.: Nonlinear coherent states. Phys. Rev. A **54**, 4560–4563 (1996)
de Matos Filho, R.L., Vogel, W.: Engineering the Hamiltonian of a trapped atom. Phys. Rev. A **58**, R1661–R1664 (1998)
Dodonov, V.V., Malkin, I.A., Man'ko, V.I.: Even and odd coherent states and excitations of a singular oscillator. Physica **72**, 597–615 (1974)
Dodonov, V.V., Man'ko, V.I.: Theory of Nonclassical States of Light. Taylor and Francis, London (2003)
Eberly, J.H., Narozhny, N.B., Sanchez-Mondragon, J.J.: Periodic spontaneous collapse and revival in a simple quantum model. Phys. Rev. Lett. **44**, 1323–1326 (1980)
Faghihi, M.J., Tavassoly, M.K.: Dynamics of entropy and nonclassical properties of the state of a Λ -type three-level atom interacting with a single-mode cavity field with intensity-dependent coupling in a Kerr medium. J. Phys. B: At. Mol. Opt. Phys. (2012). doi:10.1088/0953-4075/45/3/035502
Gerry, C.C.: Proposal for generating even and odd coherent states. Opt. Commun. **91**, 247–251 (1992)
Gerry, C.C., Hach III, E.E.: Interaction of a two-level atom with an even coherent state. Phys. Lett. A **179**, 1–8 (1993)
Glauber, R.J.: Coherent and incoherent states of the radiation field. Phys. Rev. **131**, 2766–2788 (1963)
Hekmatara, H., Tavassoly, M.K.: Sub-Poissonian statistics, population inversion and entropy squeezing of two two-level atoms interacting with a single-mode binomial field: intensity-dependent coupling regime. Opt. Commun. **319**, 121–127 (2014)
Jaynes, E.T., Cummings, F.W.: Comparison of quantum and semiclassical radiation theories with application to the beam maser. Proc. IEEE **51**, 89–109 (1963)
Lee, C., Lee, J., Nha, H., Jeong, H.: Generating a Schrödinger-cat-like state via a coherent superposition of photonic operations. Phys. Rev. A (2012). doi:10.1103/PhysRevA.85.063815
Leek, P.J.: Storing quantum information in Schrödinger's cats. Science **342**, 568–569 (2013)
Mancini, S.: Even and odd nonlinear coherent states. Phys. Lett. A **233**, 291–296 (1997)
Mandel, L.: Sub-Poissonian photon statistics in resonance fluorescence. Opt. Lett. **4**, 205–207 (1979)
Mandel, L., Wolf, E.: Optical Coherence and Quantum Optics. Cambridge University Press, Cambridge (1995)
Man'ko, V.I., Marmo, G., Sudarshan, E.C.G., Zaccaria, F.: f-Oscillators and nonlinear coherent states. Phys. Scr. (1997). doi:
Man'ko, V.I., Marmo, G., Zaccaria, F.: Moyal and tomographic probability representations for f-oscillator quantum states. Phys. Scr. (2010). doi:10.1088/0031-8949/81/04/045004
Meng, X.G., Wang, J.S., Li, Y.L.: New even and odd nonlinear coherent states and their nonclassical properties. Int. J. Theor. Phys. **46**, 1205–1214 (2007)

Mortezapour, A., Mahmoudi, M., Khajehpour, M.R.H.: Atom–photon, two-mode entanglement and two-mode squeezing in the presence of cross-Kerr nonlinearity. Opt. Quantum Electron. **47**, 2311–2329 (2015)

Moya-Cessa, H., Vidiella-Baranco, A.: Interaction of squeezed light with two-level atoms. J. Mod. Opt. **39**, 2481–2499 (1992)

Naderali, R., Motiei, H., Jafari, A.: Creation of entangled W states of four two-level atoms in a cavity via quadrapod adiabatic passage. Opt. Quantum Electron. **45**, 97–103 (2013)

Ourjoumtsev, A., Jeong, H., Tualle-Brouri, R., Grangier, P.: Generation of optical Schrödinger cats from photon number states. Nature **448**, 784–786 (2007)

Panahi, H., Asghari Rad, S.: Two and k-photon Jaynes–Cummings models and Dirac oscillator problem in Bargmann–Segal representation. Int. J. Theor. Phys. **52**, 4068–4073 (2013)

Recamier, J., Jauregui, R.: Construction of even and odd combinations of Morse-like coherent states. J. Opt. B: Quantum Semiclass. Opt. **5**, S365–S370 (2003)

Recamier, J., Gorayeb, M., Mochan, W.L., Paz, J.L.: Nonlinear coherent states and some of their properties. Int. J. Theor. Phys. **47**, 673–683 (2008)

Rempe, G., Walther, H., Klein, N.: Observation of quantum collapse and revival in a one-atom maser. Phys. Rev. Lett. **58**, 353–356 (1987)

Roknizadeh, R., Tavassoly, M.K.: The construction of some important classes of generalized coherent states: the nonlinear coherent states method. J. Phys. A: Math. Gen. **37**, 8111–8127 (2004)

Roman-Ancheyta, R., de los Santos-Sanches, O., Recamier, J.: Ladder operators and coherent states for nonlinear potentials. J. Phys. A: Math. Theor. (2011). doi:10.1088/1751-8113/44/43/435304

Roman-Ancheyta, R., Gonzales Gutierrez, C., Recamier, J.: Photon-added nonlinear coherent states for a one-mode field in a Kerr medium. J. Opt. Soc. Am. B. **31**, 38–44 (2014)

Schleish, W., Pernigo, M., Le Kien, Fam: Nonclassical state from two pseudoclassical states. Phys. Rev. A **44**, 2172–2187 (1991)

Scully, M.O., Zubairy, M.S.: Quantum Optics. Cambridge University Press, Cambridge (2001)

Shore, B.W., Knight, P.L.: The Jaynes–Cummings model. J. Mod. Opt. **40**, 1195–1238 (1993)

Sivakumar, S.: Even and odd nonlinear coherent states. Phys. Lett. A **250**, 257–262 (1998)

Sivakumar, S.: Generation of even and odd nonlinear coherent states. J. Phys. A **33**, 2289–2297 (2000a)

Sivakumar, S.: Studies on nonlinear coherent states. J. Opt. B: Quantum Semiclass. Opt. **2**, 61–75 (2000b)

Sivakumar, S.: Nonlinear Jaynes–Cummings model of atom–field interaction. Int. J. Theor. Phys. **43**, 2405–2421 (2004)

Skrypnyk, T.: The n-level, n-1-mode Jaynes–Cummings model: spectrum and eigenvectors. J. Phys. A: Math. Theor. (2013). doi:10.1088/1751-8113/46/5/052001

Tavis, M., Cummings, F.W.: Exact solution for an N-moleculeradiation-field Hamiltonian. Phys. Rev. **170**, 379–384 (1968)

Velastakis, B., Kirchmair, G., Leghtas, Z., Nigg, S.E., Frunzio, L., Girvin, S.M., Mirrahimi, M., Devoret, M.H., Schoelkopf, R.J.: Deterministically encoding quantum information using 100-photon Schrödinger cat states. Science **342**, 607–610 (2013)

Vidiella-Baranco, A., Moya-Cessa, H., Buzek, V.: Interaction of superpositions of coherent states of light with two-level atoms. J. Mod. Opt. **39**, 1441–1459 (1992)

Vidiella-Baranco, A., Moya-Cessa, H.: On the interaction of two-level atoms with superpositions of coherent states of light. J. Mod. Opt. **42**, 1547–1552 (1995)

Vidiella-Baranco, A., Moya-Cessa, H.: Superpositions of squeezed states and their interaction with two-level atoms Brazilian. J. Phys. **25**, 44–53 (1995)

Walentowitz, S., Vogel, W.: Quantum-mechanical counterpart of nonlinear optics. Phys. Rev. A **55**, 4438–4442 (1997)

Werner, R.F.: Quantum Information: An Introduction to Basic Theoretical Concepts and Experiments. Springer Tracts in Modern Physics, vol. 173. Springer, Heidelberg (2001)

Yurke, B., Stoler, D.: Generating quantum mechanical superpositions of macroscopically distinguishable states via amplitude dispersion. Phys. Rev. Lett. **57**, 13–16 (1986)

Zait, R.A.: Nonclassical statistical properties of a three-level atom interacting with a single-mode field in a Kerr medium with intensity dependent coupling. Phys. Lett. A **319**, 461–474 (2003)

Zeng, R., Ahmad, M.A., Liu, S.: Nonclassical state via superposition of two coherent states ($\frac{\pi}{2}$ out of phase) and related entangled states. Opt. Commun. **271**, 162–168 (2007)