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Four-photon nonlinear coherent states

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ABSTRACT

We have introduced the notion of 'four-photon nonlinear coherent states' (FPNCSs) as right eigenstates of the fourth power of the generalized annihilation operator A^4. It has been shown that there are four possible sets of such states which all can be expressed as the deformed Schrödinger cat type states derived from the superposition of four nonlinear coherent states (NCSs) which are $\pi/2$ out of phase. The nonclassical properties of the FPNCSs would be studied for two well-known quantum systems, a Kerr-like medium and a trapped laser-driven ion far from the Lamb-Dicke regime. We have found that the depth or the domain of the nonclassical features of FPNCSs in terms of sub-Poissonian photon statistics, higher order squeezing as well as amplitude-squared squeezing are higher than the corresponding standard states. Specifically, the proposed states in this paper exhibit sub-Poissonian statistics over an extensive range of real amplitudes which is in contrast with the lower level of sub-Poissonian characteristics of four-photon standard coherent states. Also the depth or domain of the nonclassical features of the FPNCSs can be modified by changing the magnitude of the nonlinearity parameters. Analysis of the quasi-probability distributions like Glauber–Sudarshan P-function, Husimi Q-function and Wigner distribution function verifies the nonclassical nature of the presented states. Studying the cavity field evolution in the interaction of a single two-level atom with an even NCS in the framework of f-deformed Jaynes-Cummings model shows that such a system can potentially be considered as a possible source for generating the proposed states.

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1. Introduction

The notion of standard coherent states (CSs) which was introduced by Glauber (*1*) as the eigenstates of the annihilation operator $\hat{a}$, have numerous applications in various branches of modern physics (*2*–*4*). From then until recent years, much attention has been paid to the generalizations of these states because unlike the standard CSs, many of these resulting states may exhibit some important nonclassical properties like sub-Poissonian statistics and various orders of squeezing. One of the most important of generalizations is based on multiphoton extensions of the CSs. For instance, by studying the multiphoton generalization of the squeeze operator to higher order, it is shown by Fisher et al. (*5*) that the k-photon CSs do not exist for $k > 2$ in the Hilbert space of a simple harmonic oscillator due to divergence difficulties. D' Ariano et al. (*6*, *7*) presented an alternative method of obtaining multiphoton CSs using Brandt-Greenberg multiphoton operators. In addition, they showed that these states can exhibit ordinary squeezing for special value $k = 2$. In (*8*), the authors introduced an alternative definition of the generalized k-photon CSs as the eigenvector of the operator $\hat{a}^k$ which is different from the approach in (*5*). It was found that such states indicate kth-order squeezing for $k \geq 2$. The same authors in (*9*) defined the amplitude kth-power squeezing as a new nonclassical effect and showed that k-photon CSs exhibit such nonclassical properties due to the interactions with nonlinear nonabsorbing mediums which can be modelled as an anharmonic oscillator. Also the nonclassical properties of the orthonormalised eigenstates of the higher powers of the annihilation operator ($\hat{a}^k$) for $k \geq 3$ were studied in (*10*). As a particular case of the k-photon CSs, a detailed discussion of the four-photon standard coherent states (FPSCSs) was presented in (*11*) in which it was shown that the eigenstates of the operator $\hat{a}^4$ can indicate some important nonclassical effects like photon antibunching, the higher-order squeezing as well as amplitude-squared squeezing. Considering the importance of the notion of 'quantum superposition' as a fundamental concept in quantum mechanics, following the study of multiphoton CSs, it was found that there is a significant relation between such states and the quantum superposition of CSs (*11*, *12*).

It is straightforward to show that two-photon CSs ($k = 2$) are either 'even' Schrödinger cat states or 'odd' ones (*13*). Such states as the eigenstates of $\hat{a}^2$ are respectively symmetric and anti-symmetric superpositions

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of standard CSs $|\alpha\rangle$ and $|-\alpha\rangle$. In (11), it is shown that the four-photon eigenstates of $\hat{a}^4$ can be constructed as superpositions of the four standard CSs $|\pm\alpha\rangle$ and $|\pm i\alpha\rangle$ which are $\pi/2$ out of phase with respect to each other.

On the other hand, another type of generalization of standard CSs is the notion of 'nonlinear coherent states' (NCSs) (14). This type of extension can be performed by defining the generalized annihilation operator $\hat{A} = \hat{a}f(\hat{n})$, where $f(\hat{n})$ is an operator-valued nonlinearity function of the number operator $\hat{n} = \hat{a}^\dagger\hat{a}$ and characterizes the nonlinear nature of the system (15, 16). It has been shown that NCSs which are defined as the right eigenstates of $\hat{A}$, unlike the standard CSs, indicate important nonclassical features (17–19). It is found that the 'superpositions of NCSs' obtained through the extension of the corresponding standard superposed states to the nonlinear states, might exhibit some nonclassical properties which are not present in their individual components i.e. NCSs (20–22). For example, the superposition of two NCSs, $\pi/2$ out of phase, was presented by one of us in (21) and it was shown that depending on the form of $f(\hat{n})$, these states can demonstrate some obvious nonclassical properties such as oscillatory photon count probability, sub-Poissonian statistics, amplitude-squared squeezing and the negativity of Wigner distribution function. 'Even' and 'odd' NCSs which were introduced in (23, 24) represent another type of superposition of NCSs. The explicit form of such states is given by:

$$|\alpha,f\rangle_\pm = N_\pm(|\alpha,f\rangle \pm |-\alpha,f\rangle) \quad (1)$$

where $|\alpha,f\rangle$ is any class of NCSs, $(+)$ and $(-)$ are respectively related to even and odd NCSs and $N_\pm$ is the normalization factor. The important characteristic of the even and odd NCSs is that they are eigenstates of the square of the deformed annihilation operator, $\hat{A}^2|\alpha,f\rangle_\pm = \alpha^2|\alpha,f\rangle_\pm$. Thus, these states which are obviously Schrödinger cat states of the deformed fields, can be considered 'two-photon NCSs'.

In this paper, we introduce the 'four-photon nonlinear coherent states' (FPNCSs) as an extension of the two-photon eigenstates of $\hat{A}^2$ to the four-photon eigenstates of $\hat{A}^4$. These formalism is also equivalent to the extension of eigenstates of $\hat{a}^4$ (FPSCSs) to the corresponding nonlinear case i.e. the eigenstates of $\hat{A}^4$ (FPNCSs). Based on the lowest number of photons (the lowest number state which appear in the Fock space representation), there exist four sets of such states. A prominent feature of these states is that they are unique superpositions of NCSs $|\pm\alpha,f\rangle$ and $|\pm i\alpha,f\rangle$. We apply our formalism to two well-known quantum systems through the relevant nonlinearity functions and investigate the nonclassical features of the proposed states such as oscillatory photon count probability, sub-Poissonian statistics, Nth-order squeezing as well as amplitude-squared squeezing. In all cases, we compare FPNCSs with the corresponding standard states (FPSCSs) which are simply obtainable by setting $f(\hat{n}) = 1$ in our formalism. Also the quasi-probability distribution of the introduced states will be studied via the Glauber–Sudarshan P-function and the Husimi Q-function. In the following, the negativity of the Wigner distribution function as an important nonclassical property of FPNCSs will be established manifestly.

There has been significant interest in the generation of various quantum field states, especially the superposition of CSs, via the interaction of atomic systems with the cavity fields (25, 26). The standard model for describing the interaction of a single-mode of the quantized electromagnetic field with a two-level atom was introduced by Jaynes and Cummings (27). Also the study of the interaction of the two-level atoms with the superposition of CSs through the standard Jaynes-Cummings model (JCM) has been of general interest (28, 29). The standard JCM as an exactly solvable model has been extended in different directions in the past fifty years (30–34) because of its potential in developing theoretical results which agree with experimental observations. Lately, a special type of the generalized JCM has been introduced in (35) using the framework of an f-deformed oscillator (15). The most important characteristic of this model is that it allows one to study the interaction of NCSs and their evolution with a two-level atom, because the medium has been modelled by a deformed oscillator instead of the standard harmonic oscillator. Finally, based on the f-deformed JCM, we study the interaction of a single two-level atom with the cavity field when initially prepared in an even nonlinear coherent state (ENCS) (36) and show how the field states at the collapses time of atomic inversion are to be considered as a superposition of NCSs $\pi/2$ out of phase with respect to each other. Although these superpositions are not necessarily FPNCSs, the similarity of the corresponding Q-function with the four peak structure of the field in this physical system indicates that such a system is a potential candidate for generating FPNCSs.

The paper is organized as follows: in the next section, the FPNCSs are introduced as the right eigenstates of $\hat{A}^4$. In Section 3, the photon statistics of these states is investigated through the photon count probability and the second-order correlation function. The squeezing properties of the presented states have been studied in Section 4. The singularity of the Glauber–Sudarshan P-function of the introduced states is shown in Section 5. Additionally, the Husimi Q-function structure in the phase space as well as the negativity of FPNCSs Wigner function are illustrated in this section. In Section 6, the generation of the superposition of NCSs $\pi/2$ out of phase, is presented via the interaction of a single two-level atom with an ENCS in the framework of the f-deformed JCM. Finally, a summary containing the results is provided in Section 7.

2. Theory

The NCSs method is based on the deformation of standard annihilation and creation operators with an intensity-dependent function $f(\hat{n})$, according to the relations $\hat{A} = \hat{a}f(\hat{n})$ and $\hat{A}^\dagger = f^\dagger(\hat{n})\hat{a}^\dagger$ where $\hat{a}$, $\hat{a}^\dagger$ and $\hat{n} = \hat{a}^\dagger\hat{a}$ are respectively bosonic annihilation, creation and number operators. The $f(\hat{n})$ is an operator-valued function of the number operator which is assumed to be real and nonnegative, in this paper. The commutators between $\hat{A}$ and $\hat{A}^\dagger$ read as $[\hat{A},\hat{A}^\dagger] = (\hat{n}+1)f^2(\hat{n}+1) - \hat{n}f^2(\hat{n})$ and the NCSs satisfy the typical eigenvalue equation:

$$\hat{A}|\alpha,f\rangle = \alpha|\alpha,f\rangle, \qquad \alpha \in \mathbb{C}. \tag{2}$$

The Fock space representation of these states is explicitly expressed by:

$$|\alpha,f\rangle = N_f \sum_{n=0}^{\infty} \frac{\alpha^n}{f(n)!\,\sqrt{n!}}\,|n\rangle, \tag{3}$$

where N_f is some appropriate normalization factor determined from $\langle\alpha,f|\alpha,f\rangle = 1$, and by convention $f(n)! \doteq f(1)f(2)\ldots f(n)$ and $f(0)! \doteq 1$.

Now we define FPNCSs as the eigenstates of the operator $\hat{A}^4$:

$$\hat{A}^4|\xi^{(l)},f\rangle = \xi|\xi^{(l)},f\rangle. \tag{4}$$

Here, ξ is a complex number and $l = 0,1,2,3$ representing the lowest number state which appears in the superposition. One can obtain the following expansion of the FPNCSs in terms of the number states:

$$|\xi^{(l)},f\rangle = \sum_{n=0}^{\infty} C_{4n+l}|4n+l\rangle, \qquad l = 0,1,2,3. \tag{5}$$

By substituting relation (5) in (4), we arrive at the explicit and general form of FPNCSs in the number basis as:

$$|\xi^{(l)},f\rangle = N_l \sum_{n=0}^{\infty} \frac{\xi^n}{f(4n+l)!\sqrt{(4n+l)!}}|4n+l\rangle, \qquad l = 0,1,2,3. \tag{6}$$

Using $\langle\xi^{(l)},f|\xi^{(l)},f\rangle = 1$, one finds the normalization factor as:

$$N_l = \left(\sum_{n=0}^{\infty} \frac{|\xi|^{2n}}{[f(4n+l)!]^2(4n+l)!}\right)^{\frac{-1}{2}}. \tag{7}$$

Based on the minimum number of photons in the proposed states (6) which is determined by the value of 'l', we realize that there are four distinct categories of FPNCSs which might demonstrate different nonclassical properties. It is easy to show that the two FPNCSs, corresponding to different values of 'l', are orthogonal:

$$\langle\xi^{(l)},f|\xi^{(l')},f\rangle = \delta_{l,l'}. \tag{8}$$

Based on the Man'ko et al. formalism in (*15*), there is a well-known relation between the expansion coefficients C_n's of the NCSs and the corresponding nonlinearity function in the form $f(n) = C_{n-1}/\sqrt{n}C_n$. This would help us to recognize the nonlinearity of a generalized CS. Using this relation, we found that there does not exist a well-defined nonlinearity function corresponding to the states in (6). In other words, the proposed states in (6) could not be classified in the 'family of NCSs' or equivalently they could not be reproduced as the right eigenstate of a new deformed annihilation operator.

Furthermore, it is clear that the constructed FPNCSs are in fact superpositions of $|\pm\alpha,f\rangle$ and $|\pm i\alpha,f\rangle$. However, even and odd NCSs (two-photon NCSs) result from an interference of the states $|\alpha,f\rangle$ and $|-\alpha,f\rangle$. For instance, the first class of FPNCSs corresponding to $l = 0$ in (6), i.e. $|\xi^{(0)},f\rangle$, could be generated by the following superposition of four-component NCSs $\pi/2$ out of phase:

$$|\xi^{(0)},f\rangle = \mathcal{N}(|\alpha,f\rangle + |-\alpha,f\rangle + |i\alpha,f\rangle + |-i\alpha,f\rangle). \tag{9}$$

Here, $\mathcal{N}$ is the normalization factor and $\xi^{(0)} = \alpha^4$. It is clear that FPNCSs are 'f-deformed Schrödinger cat states' composed of four macroscopic components.

For the physical realization, we will apply the presented formalism to generalized CSs using two known quantum systems with their relevant nonlinearity functions. For the first one, the following real nonlinearity function is used (*35*, *37*):

$$f_j(\hat{n}) = \sqrt{1 - \frac{\chi}{\omega_0}(1 - \hat{n}^{j-1})}, \tag{10}$$

where ω_0 is the field frequency, χ denotes the real anharmonicity parameter related to the optical properties of the Kerr medium with $0 \le \chi \ll \omega_0$ and j can take the discrete values $1, 2, 3 \ldots$. As a physical result of using deformation function (10), one can easily find that by setting (10) instead of nonlinearity function in 'normal-ordered' of the Man'ko et al. Hamiltonian $\hat{H}_M = \frac{\hbar\omega_0}{2}(\hat{A}^\dagger\hat{A}+\hat{A}\hat{A}^\dagger)$ related to the f-deformed oscillator (38, 39), the deformed free field comes in the form below:

$$\hat{H}_M = \hbar\nu\hat{n} + \hbar\chi\hat{n}^j, \tag{11}$$

where $\nu = \omega_0 - \chi$. As a special case, we can recover the Kerr Hamiltonian (40) for $j = 2$:

$$\hat{H}_{\text{Kerr}} = \hbar\omega_0\hat{a}^\dagger\hat{a} + \hbar\chi\hat{a}^{\dagger 2}\hat{a}^2, \tag{12}$$

Notify that the Hamiltonian (11) can be considered as a special case of the anharmonic oscillator Hamiltonian $\hat{H} = \omega_0\hat{n} + \chi\hat{n}^j$ studied by Yurke and Stoler in (41). They demonstrated that under the evolution of this Hamiltonian, an initial standard CS $|\alpha\rangle$ will evolve into a coherent superposition of a finite number of CSs that are macroscopically distinguishable when $|\alpha|$ is large enough. Therefore, the explicit form of FPNCSs in a Kerr-like medium can be constructed by applying (10) with $j = 2$ in (6) as:

$$|\xi^{(l)}, f\rangle_{\text{Kerr}} = N_l^{(\gamma)} \sum_{n=0}^{\infty} \frac{\xi^n}{(1-1/4\gamma)\sqrt{(4n+l)!\,(1/4\gamma)^{4n+l}\,(4\gamma)_{4n+l}}} \times |4n+l\rangle, \qquad l = 0,1,2,3 \tag{13}$$

with the normalization factor:

$$N_l^{(\gamma)} = \sqrt{\frac{l!\,(1/4\gamma)^l\,(1-1/4\gamma)\,(4\gamma)_l}{{}_1F_8(1;\kappa_1,\kappa_2,\kappa_3,\kappa_4,\vartheta_0,\vartheta_1,\vartheta_2,\vartheta_3;(\gamma/4)^4|\xi|^2)}}, \tag{14}$$

where $\gamma = \omega_0/4\chi$, $\kappa_i = (l+i)/4$ and $\vartheta_i = \kappa_i + \gamma$. Also ${}_pF_q(a;b;x) = {}_pF_q(\{a_1,\ldots,a_p\};\{b_1,\ldots,b_q\};x)$ is the generalized hypergeometric function and $(a)_n = \Gamma(a+n)/\Gamma(a)$ is the Pochhammer symbol. One can easily recover FPSCSs by setting $\chi = 0$ in above expressions. From now on, we would only consider $j = 2$ and in the rest of the paper, 'Kerr nonlinearity function' is a reference to the deformation function (10).

For the second system, we use the following nonlinearity function (14):

$$f(n) = \frac{L_n^1(\eta^2)}{(n+1)\,L_n^0(\eta^2)}, \tag{15}$$

where $L_n^m(x)$ are generalized Laguerre polynomials, η is known as the Lamb-Dicke parameter which is a measure of nonlinearity. The above function is naturally arisen from the NCSs related to the center-of-mass motion of a trapped ion (42). In fact, the nonlinearity described with (15) may be generated in a trapped and bichromatically laser-driven ion far from the Lamb-Dicke regime. In the Lamb-Dicke regime where $\eta \ll 1$ considering that $L_n^m(\eta^2 \ll 1) \simeq \frac{(n+m)!}{n!\,m!}$, the nonlinearity function reduces to $f(n) \simeq 1$ corresponding to FPSCSs. In all of the latter references, we call (15) 'trapped ion nonlinearity function'.

3. Photon statistics

The probability of finding 'n' photons in the field for the all sets of FPNCSs is given by:

$$P_l(n) = |\langle n|\xi^{(l)}, f\rangle|^2 = \frac{|N_l|^2\ |\xi|^{(n-l)/2}}{[f(n)!]^2\ n!}, \qquad n = l,\ l+4,\ l+8,\ldots, \tag{16}$$

and for other values of n, $P_l(n) = 0$. From this expression, it is evident that FPNCSs have strongly oscillating behaviour which is a peculiarity of highly nonclassical states. One should be aware that although the oscillatory photon count probability (as well as other nonclassical properties) of NCSs depends on the form of nonlinearity function, this nonclassicality feature is a natural signature of our FPNCSs. The oscillatory distribution probability of the states (6) with $l = 0$ has been shown in Figure 1(a) for Kerr nonlinearity function (10), $|\xi| = 25$ and $\chi/\omega_0 = 0.1$. It is clear that such oscillations may be generated by the quantum interference effects in (9). It should be mentioned that although such oscillations are also present in the nondeformed FPSCSs (11), in FPNCSs, the profile of the photon count probability will be determined by $f(n)$. The same oscillatory behaviour exists for other values of 'l' as well as the other nonlinearity function (15). Also, we found that for larger values of $|\xi|$, the probability of finding the higher numbers of photons, 'n', in the field, would increase.

To investigate the sub-Poissonian statistical behaviour (as a nonclassical feature) of FPNCSs, we would consider the second-order correlation function (43):

$$g_l^{(2)}(0) = \frac{\langle\hat{n}^2\rangle_l - \langle\hat{n}\rangle_l}{\langle\hat{n}\rangle_l^{\,2}}, \tag{17}$$

where index 'l' in above expression shows that all the expectation values is taken between FPNCSs in (6). It is worth mentioning that the second-order correlation function can be measured by a set of two detectors (44), for example, the standard Hanbury–Brown–Twiss coincidence arrangement. Please note that in general, depending on the particular form of $f(n)$, a quantum state may exhibit super-Poissonian, Poissonian or sub-Poissonian, respectively if $g^{(2)}(0) > 1$ (bunching effect), $g^{(2)}(0) = 1$ or $g^{(2)}(0) < 1$ (antibunching effect) and the states that possess sub-Poissonian statistics are nonclassical. Considering (6), the expectation values of the 'm' power of the photon number operator $\hat{n}$ are given by:

$$\langle \hat{n}^m \rangle_l = |N_l|^2 \sum_{n=0}^{\infty} |\mathcal{Q}_{n,l}|^2 (4n + l)^m, \tag{18}$$

where:

$$\mathcal{Q}_{n,l} = \frac{\xi^n}{f(4n + l)!\sqrt{(4n + l)!}}, \tag{19}$$

In Figure 1(b), we have plotted the second-order correlation function $g_l^{(2)}(0)$ against $|\xi|$,where $\xi = |\xi| \exp(i\varphi)$ and the φ is chosen to be zero. We consider $f(n) = 1$, i.e. the FPSCSs (solid line) and FPNCSs corresponding to Kerr nonlinearity function (dot-dashed line) and trapped ion nonlinearity function (dashed line). The parameters are $\chi/\omega_0 = 0.2$ and $\eta = 0.2$ for (10) and (15) respectively and in all cases, we consider the $l = 0$. It is obvious that unlike FPSCSs, both types of FPNCSs have sub-Poissonian statistics in a wide range of $|\xi|$. Although the FPSCSs can also exhibit sub-Poissonian behaviour in a small range of $|\xi|$, for $|\xi| \gtrsim 35$, we have $g_l^{(2)}(0) = 1$; this means that such states have Poissonian distribution just like the standard CSs $|\alpha\rangle$. In addition, we found that, for other values of l, while the FPSCSs behave like the standard CSs (Poissonian distribution) in a wide range of $|\xi|$, the corresponding nonlinear states are nonclassical (sub-Poissonian distribution). Furthermore, as a special case, we found that for $l = 3$, in contrast to FPSCSs, both types of FPNCSs exhibit sub-Poissonian statistics over the whole range of $|\xi|$. Also it turns out that increasing the parameters χ/ω_0 and η which are measures of nonlinearity would decrease the amount of second-order correlation function or equivalently increase the depth of nonclassicality feature.

4. Squeezing properties

Investigating the squeezing properties of electromagnetic fields has become one of the major objectives of quantum optics due to the wide range of its applications, e.g. in interferometric techniques (*45*, *46*), precision measurements (*47*, *48*) and optical communication networks (*49*, *50*). This phenomenon is indicated by decreasing the quantum fluctuations in one field quadrature compared to a CS with an increase in the conjugate quadrature such that the uncertainty relation between the canonically conjugate variances be satisfied. There are several ways to define the squeezing parameter. For example, the normal squeezing (second-order squeezing), higher order squeezing as well as amplitude-squared squeezing can be generally defined by introducing the following Hermitian operators:

$$\hat{X}_k = \frac{\hat{a}^k + \hat{a}^{\dagger k}}{2}, \qquad \hat{Y}_k = \frac{\hat{a}^k - \hat{a}^{\dagger k}}{2i}. \tag{20}$$

A state is said to be squeezed in the Nth-order (N is even) if $\langle (\Delta\hat{x}_i)^N \rangle$ ($\hat{x}_i = \hat{X}_1$ or $\hat{Y}_1$) is smaller than its corresponding CS value. Here the quadrature operators $\hat{X}_1$ and $\hat{Y}_1$ are obtained by setting $k = 1$ in Equation (20). For measuring the degree of squeezing, we can use the squeezing parameter $q^{(N)}$ which has been introduced by Hong and Mandel (*51*):

$$q^{(N)} = \frac{\langle (\Delta\hat{x}_i)^N \rangle - (N-1)!!C^{N/2}}{(N-1)!!C^{N/2}}, \tag{21}$$

with $C = 1/4$ in our case. The Nth-order squeezing condition is:

$$-1 \le q^{(N)} < 0, \tag{22}$$

and the maximum squeezing is achieved when $q^{(N)} = -1$.

Considering Equation (6), the non-zero expectation values of the powers of the annihilation operator can be expressed in the general form:

$$\langle \hat{a}^{4m} \rangle_l = |N_l|^2 \sum_{n=0}^{\infty} \mathcal{Q}^*_{n,l} \mathcal{Q}_{n+m,l} \sqrt{\frac{(4(n+m)+l)!}{(4n+l)!}}, \quad m = 1, 2, 3, \ldots, \tag{23}$$

and note that $\langle \hat{a}^{\dagger 4m} \rangle_l = \langle \hat{a}^{4m} \rangle_l^*$. Considering that $\langle \hat{a} \rangle_l = \langle \hat{a}^2 \rangle_l = 0$ for all values of l, the normal squeezing of FPNCSs would be zero just like FPSCSs.

The higher order squeezing ($N \ge 4$) of a quantum field, characterized by the parameter $q^{(N)}$ in Equation (21), can be considered as a natural generalization of the normal squeezing (usual second-order squeezing). In order to determine the degree of higher order squeezing, we consider the 4th-order and 6th-order squeezing corresponding to $N = 4$ and $N = 6$ in Equation (21). The respective squeezing conditions could be obtained by Equation (22), in terms of:

$$q_l^{(4)} = \frac{1}{3}(\langle \hat{a}^4 \rangle_l + \langle \hat{a}^{\dagger 4} \rangle_l + 6\langle \hat{n}^2 \rangle_l + 6\langle \hat{n} \rangle_l), \tag{24}$$

$$q_l^{(6)} = \frac{1}{15}(15\langle \hat{a}^4 \rangle_l + 15\langle \hat{a}^{\dagger 4} \rangle_l + 6\langle \hat{a}^\dagger \hat{a}^5 \rangle_l + 6\langle \hat{a}^{\dagger 5} \hat{a} \rangle_l + 20\langle \hat{n}^3 \rangle_l + 30\langle \hat{n}^2 \rangle_l + 40\langle \hat{n} \rangle_l), \tag{25}$$

where only non-zero expectation values have been explicitly expressed and they have been calculated in terms of normally ordered functions of $\hat{a}$ and $\hat{a}^\dagger$. Also $\langle \hat{n}^m \rangle_l$ and $\langle \hat{a}^{4m} \rangle_l$ are respectively given by Equations (18) and (23) and we have:

$$\langle \hat{a}^\dagger \hat{a}^5 \rangle_l = |N_l|^2 \sum_{n=0}^{\infty} \mathcal{Q}^*_{n,l} \mathcal{Q}_{n+1,l}\, (4n + l) \sqrt{\frac{(4n + l + 4)!}{(4n + l)!}}. \tag{26}$$

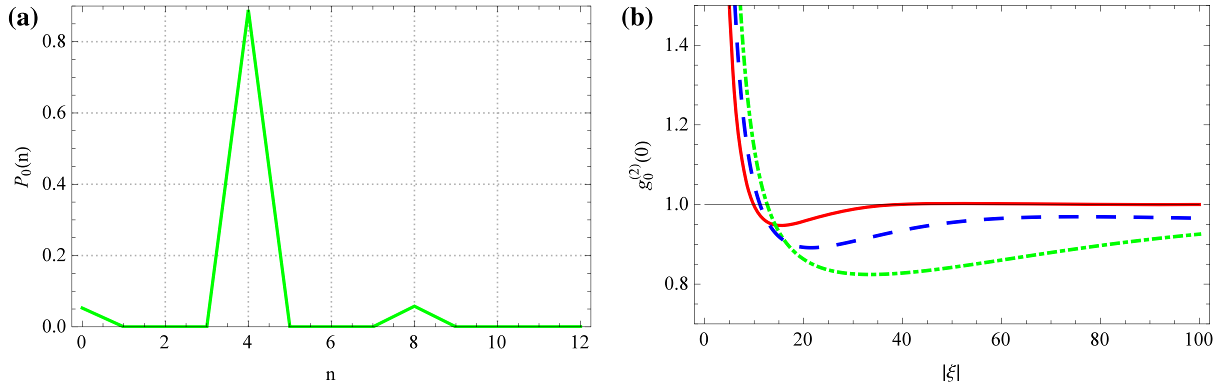

Figure 1. (a) The probability distribution $P_l(n)$ against n for states (6) with Kerr nonlinearity function (10) and $l = 0$, $\chi/\omega_0 = 0.1$, $|\xi| = 25$. (b) The second-order correlation function $g_l^{(2)}(0)$ against $|\xi|$ for states (6) with $f(n) = 1$ (solid line), the Kerr nonlinearity function (10) with $\chi/\omega_0 = 0.2$ (dot-dashed line) and the trapped ion nonlinearity function (15) with $\eta = 0.2$ (dashed line). In all cases, we consider $l = 0$ and $\varphi = 0$.

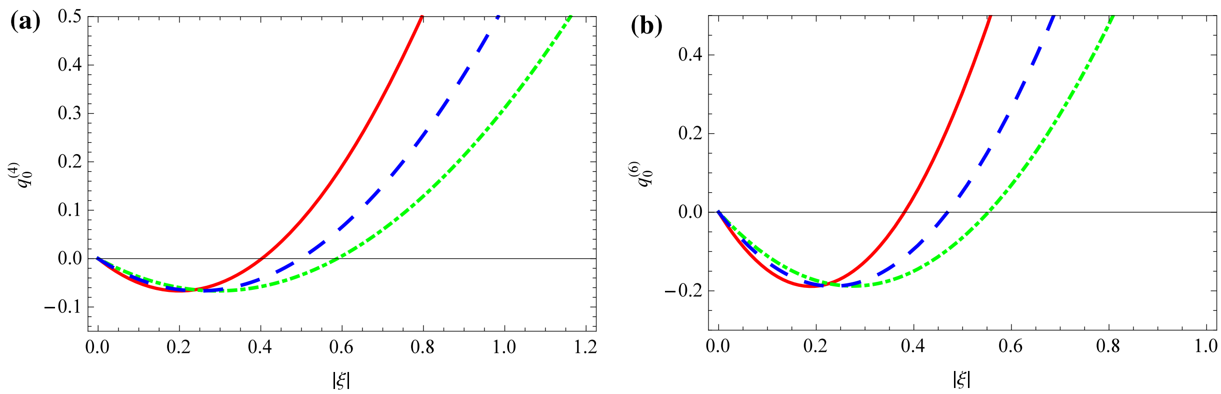

Figure 2. The same as Figure 1(b) but for (a) $q_l^{(4)}$ (b) $q_l^{(6)}$ and $\varphi = \pi$.

Please note that $\langle\hat{a}^{\dagger 5}\hat{a}\rangle_l = \langle\hat{a}^{\dagger}\hat{a}^5\rangle_l^*$. The relations of $q_l^{(4)}$ and $q_l^{(6)}$ with respect to $|\xi|$ are presented in Figure 2(a) and (b) where the parameters taken in Figure 1(b) have been used here as well, except for $\varphi = \pi$ which minimizes the amount of squeezing. One can easily see that the domain of the higher order squeezing (the domain of the nonclassicality feature) of FPNCSs is obviously higher than FPSCSs (the dashed and dot-dashed curves in comparison to the solid curve). It was also found that the squeezing range of the nonlinear states would increase by increments of χ/ω_0 and η, this means that, unlike the standard states, the domain of the nonclassicality of FPNCSs can be controlled by changing the nonlinearity parameters. Comparing Figure 2(a) and (b), one would find that the amount of squeezing (the depth of the nonclassicality feature) is directly related to the squeezing order N. It should be mentioned that although we found that the higher order squeezing occur only for $l = 0$ i.e. the $|\xi^{(0)}, f\rangle$, the value of squeezing parameter $q_l^{(N)}$ for other sets of nonlinear states is always lower than the standard ones.

Another important nonclassicality feature of FPNCSs is the amplitude-squared squeezing. In this context, we found that in specific regions of $|\xi|$ which are slightly higher than the corresponding regions for FPSCSs, this nonclassicality mechanism would induced some distinguishable effect.

5. Quasi-probability distributions

5.1. *Glauber–Sudarshan P-function*

The Glauber–Sudarshan P-function (or simply P-function) is referred to as the Glauber–Sudarshan P-representation or equivalently CS representation (*1*, *54*). The name CS representation implies that the density operator $\hat{\rho}$ is expressed in terms of the diagonal pair CSs $|\alpha\rangle\langle\alpha|$ as:

$$\hat{\rho} = \int P(\alpha, \alpha^*)|\alpha\rangle\langle\alpha| \mathrm{d}^2\alpha \tag{27}$$

The Glauber–Sudarshan P-function can be considered as the most meaningful criterion for the nonclassicality of the quantum states. Whenever it becomes nega-

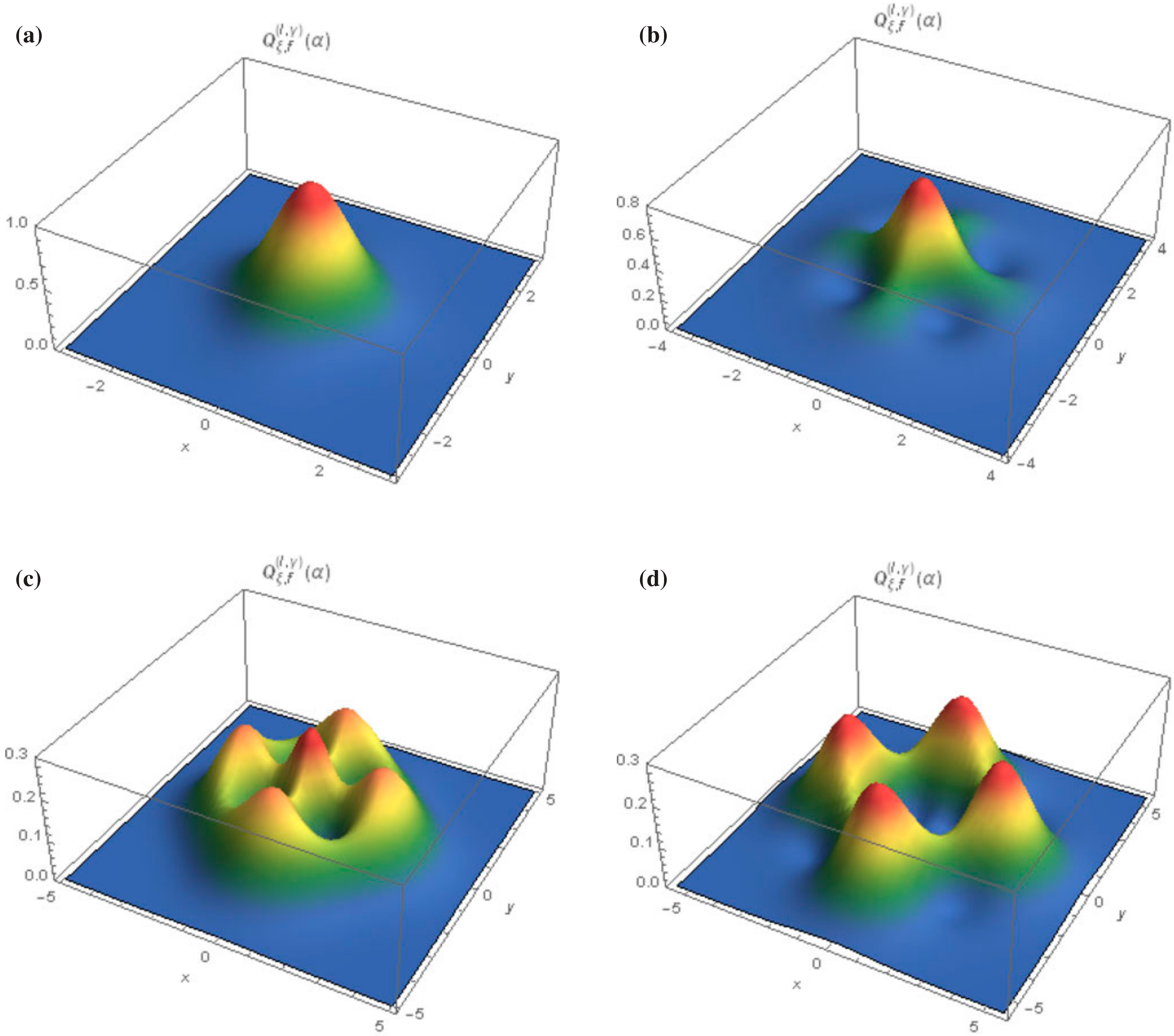

Figure 3. The Husimi Q-function $Q^{(l,\gamma)}_{\xi,f}(\alpha)$ of FPNCSs against $x = Re(\alpha)$ and $y = Im(\alpha)$ for Kerr nonlinearity function (10) with $l = 0$, $\chi/\omega_0 = 0.1$ and (a) $|\xi| = 0.5$, (b) $|\xi| = 4$, (c) $|\xi| = 10$, (d) $|\xi| = 50$.

tive or highly singular (more singular than a delta function), the corresponding quantum states are called nonclassical. The more useful integral formulas for $P(\alpha,\alpha^*)$ can be obtained using the inverse Fourier transform of $\langle -\beta|\hat{\rho}|\beta\rangle e^{|\beta|^2}$ as (58):

$$P(\alpha,\alpha^*) = \frac{e^{|\alpha|^2}}{\pi^2}\int \langle -\beta|\hat{\rho}|\beta\rangle e^{|\beta|^2} e^{(-\beta\alpha^*+\beta^*\alpha)}\mathrm{d}^2\beta, \tag{28}$$

where $|\beta\rangle$ and $|-\beta\rangle$ are ordinary CSs with β and $-\beta$ as the eigenvalues of the standard annihilation operator $\hat{a}$. Now considering the density operator of FPNCSs i.e. $\hat{\rho} = |\xi^{(l)}, f\rangle\langle\xi^{(l)}, f|$, we can find the function $P^{(l)}_{\xi,f}(\alpha,\alpha^*)$ for these states

$$P^{(l)}_{\xi,f}(\alpha,\alpha^*) = \frac{e^{|\alpha|^2}}{\pi^2}\int \langle -\beta|\xi^{(l)}, f\rangle\langle\xi^{(l)}, f|\beta\rangle \times e^{|\beta|^2} e^{(-\beta\alpha^*+\beta^*\alpha)}\mathrm{d}^2\beta. \tag{29}$$

By applying Equation (6) to the above expression, we obtain the explicit formula for FPNCSs P-function:

$$P^{(l)}_{\xi,f}(\alpha,\alpha^*) = \frac{e^{|\alpha|^2}|N_l|^2}{\pi^2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \times \frac{\xi^m\xi^{*n}}{f(4m+l)!f(4n+l)!(4m+l)!(4n+l)!} \times \frac{\partial^{4(m+n)+2l}}{\partial\alpha^{4m+l}\,\partial\alpha^{*4n+l}}\,\delta^2(\alpha), \tag{30}$$

where $\delta^2(\alpha)$ is the two-dimensional delta function. Existence of the higher order derivatives of a delta function in the $P^{(l)}_{\xi,f}(\alpha,\alpha^*)$ clearly implies a high degree of singularity or equivalently a high level of nonclassicality in the states $|\xi^{(l)},f\rangle$.

It is worth mentioning that the necessary and sufficient condition for the nonclassicality of an arbitrary bosonic quantum state has been lately formulated by introducing a general regularization method based on nonclassicality filters and nonclassicality quasi-probability distributions (*55*). The advantage of this method is that the presented regularized P-function, which is called 'nonclassicality quasi-probability', is capable of uncovering all negativities of the original P-function in terms of regular and, in general, measurable characteristic functions. Moreover, some experimental aspects of this theory have been lately approved in some later works. For example, the experimental reconstruction of a nonclassicality quasi-probability for a single-photon-added thermal state has been reported in (*56*). Also by deriving pattern functions for the direct experimental determination of the nonclassicality quasi-probabilities, the authors in (*57*) applied this method to a squeezed vacuum state of light that was generated by parametric down-conversion in a second-order nonlinear crystal. Here, we have simply extracted the P-function in order to reveal the existence of nonclassicality nature of the states under study. Further investigations are required to achieve a more detailed and subtle understanding of necessary and sufficient conditions for nonclassicality in this theory.

5.2. Husimi Q-function

The quasi-probability distributions can be used in describing the electromagnetic field modes and recovering valuable information about the quantum state of the field (*58*). To this aim, we consider the Husimi Q-function in phase space which is always positive and therefore a true probability distribution. Since we are dealing with NCSs, we manipulate the generalized form of the Husimi Q-function (*35*) as:

$$Q_f(\alpha) = \frac{1}{\pi}\left\langle \alpha,f|\hat{\rho}|\alpha,f\right\rangle, \tag{31}$$

where $\hat{\rho}$ is the density operator of the field and the average is taken over the NCSs $|\alpha,f\rangle$. When the field is a FPNCS given by (6), the generalized Husimi Q-function can be written as:

$$Q^{(l)}_{\xi,f}(\alpha) = \frac{|N_f|^2|N_l|^2}{\pi}|G^{(l)}_{\xi,f}(\alpha)|^2, \tag{32}$$

where $N_f = \left(\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{[f(n)!]^2(n)!}\right)^{-1/2}$ is the normalization factor of the NCSs in (3), N_l is given by (7) and:

$$G^{(l)}_{\xi,f}(\alpha) = \sum_{n=0}^{\infty}\frac{\alpha^{*4n+l}\xi^n}{[f(4n+l)!]^2(4n+l)!}. \tag{33}$$

For studying the Husimi Q-function of FPNCSs, in this section, we will consider the Kerr nonlinearity function (10) for which we have:

$$Q^{(l,\gamma)}_{\xi,f}(\alpha) = \frac{|N^{(\gamma)}_f|^2|N^{(\gamma)}_l|^2}{\pi} \times\left|\sum_{n=0}^{\infty}\frac{\alpha^{*4n+l}\xi^n}{(1-1/4\gamma)^2\,(4n+l)!\,(1/4\gamma)^{4n+l}\,(4\gamma)_{4n+l}}\right|^2, \tag{34}$$

where $N^{(\gamma)}_l$ is given by (14), $N^{(\gamma)}_f = \sqrt{\frac{1-1/4\gamma}{{}_0F_1(4\gamma;4\gamma|\alpha|^2)}}$ and ${}_0F_1(a;b;x)$ is confluent hypergeometric function.

In Figure 3, we display $Q^{(l,\gamma)}_{\xi,f}(\alpha)$ against imaginary and real parts of α with $l=0$, $\chi/\omega_0=0.1$ and different values of $|\xi|$. As it is clear from Figure 3(a), at low intensities, the Husimi Q-function includes a well-localized peak centred at $x=y=0$. In fact, such a Gaussian distribution corresponds to the vacuum state at $|\xi|=0$ whose maximum amount is obviously reduced as $|\xi|$ increases and simultaneously four symmetrical peaks emerge around the central peak (see Figure 3(b) and (c)). At high values of $|\xi|$, it is seen from Figure 3(d) that the central peak associated with the vacuum state completely vanishes and the four peaks symmetrically located on the x and y axes can be observed to be more apparent. It is found that the same behaviour exists for other values of 'l' as well as the trapped ion nonlinearity function (15).

5.3. Wigner distribution function

The Wigner quasi-probability distribution function is a square integrable function used for studying quantum corrections to classical statistical mechanics. This function always does exist and like the other distributions can be used to calculate averages as an essential task in quantum physics. For certain quantum states, this function may take negative values in some regions of the phase space, a situation which is classically impossible. So negativity of Wigner distribution function in the phase space authenticates the nonclassicality nature of the states (*59*, *60*). In this paper, we use the special form of Wigner function based on the work of (*61*, *62*) which is expressed in terms of the displaced number-state expectation values (*17*) as follows:

$$W(\alpha) = \frac{2}{\pi}\sum_{n=0}^{\infty}(-1)^n\left\langle \alpha,n|\hat{\rho}|\alpha,n\right\rangle, \tag{35}$$

where the displaced number-states $|\alpha, n\rangle$ are defined as (63, 64):

$$|\alpha, n\rangle = \exp\left(\frac{-|\alpha|^2}{2}\right)\sum_{k=0}^{\infty}\left(\frac{n!}{k!}\right)^{1/2}\alpha^{n-k}L_k^{n-k}(|\alpha|^2)|k\rangle, \tag{36}$$

Using (6) and for the density matrix $\hat{\rho} = |\xi^{(l)}, f\rangle\langle\xi^{(l)}, f|$, one could obtain the following general expression for Wigner distribution function of FPNCSs:

$$W_{\xi,f}^{(l)}(\alpha) = \frac{2|N_l|^2}{\pi e^{|\alpha|^2}}\sum_{n=0}^{\infty}(-1)^n|\alpha|^{2(l-n)}|\mathcal{G}_{\xi,f}^{(l)}(\alpha, n)|^2, \tag{37}$$

where:

$$\mathcal{G}_{\xi,f}^{(l)}(\alpha, n) = \sum_{s=0}^{n}\frac{(-|\alpha|^2)^s}{s!}\sum_{m=0}^{\infty}\frac{(\xi\alpha^{*4})^m\,(n!)^{1/2}}{[f(4m+l)!]\,(4m+l)!} \times \binom{4m+l}{n-s}, \tag{38}$$

and $\binom{a}{b}$ is the binomial coefficient. For the states (13) in a Kerr medium, the explicit form of the Wigner distribution function is:

$$W_{\xi,f}^{(l,\gamma)}(\alpha) = \frac{2|N_l^{(\gamma)}|^2}{\pi e^{|\alpha|^2}}\sum_{n=0}^{\infty}(-1)^n|\alpha|^{2(l-n)}\left|\sum_{s=0}^{n}\frac{(-|\alpha|^2)^s}{s!}\sum_{m=0}^{\infty} \times \frac{(\xi\alpha^{*4})^m\,(n!)^{1/2}\,\mu_{m,s,n}}{(1-1/4\gamma)\,(4m+l)!\,\sqrt{(1/4\gamma)^{4m+l}\,(4\gamma)_{4m+l}}}\right|^2, \tag{39}$$

where $\mu_{m,s,n} = \binom{4m+l}{n-s}$ and $N_l^{(\gamma)}$ is given by (14). The Wigner function $W_{\xi,f}^{(l,\gamma)}(\alpha)$ of FPNCSs in a Kerr medium for $l = 0$ is plotted against imaginary and real parts of α in Figure 4 in which all the other parameters are considered to be the same as Figure 3. As it is clear, the Wigner distribution function becomes negative at some phase space points over the whole range of $|\xi|$, therefore, one can conclude that the negativity of Wigner function is a natural signature of $|\xi^{(0)}, f\rangle$. It is notable that the Wigner function of the NCSs with various nonlinearity function are positive in all phase space similar to that of the standard CSs which is positive with a Gaussian form (21). Therefore, the negativity of Wigner distribution function for the state $|\xi^{(0)}, f\rangle$ can be attributed to the interference effects which have clearly been recognized in Equation (9). Moreover, it is evident that increasing $|\xi|$ would make the Wigner function become more negative over more extensive regions of the phase space (compare Figure 4(a) and 4(d)). We also found that although all sets of FPNCSs indicate nonclassical features, the negativity of Wigner function is more evident in cases $l = 1, 3$.

6. Superposition of four-component NCSs $\pi/2$ out of phase

The f-deformed JCM as a nonlinear JCM is constructed from the standard JCM by deforming the single-mode field operators (35). The main characteristic of this model is that it can describe the interaction of a two-level atom with the NCSs and their evolution. It is notable that this model, as a theoretical generalized atom-field interaction model, can be applied to real problems with nonlinear interactions in the field of nonlinear optics (65, 66). On the other hand, studying the interactions of CSs superpositions with matter based on the standard JCM would lead to a better understanding of superposition principle through field evolution monitoring (28, 29). In this section, we are going to study the interaction of an ENCS with a single two-level atom inside a lossless optical cavity based on the f-deformed JCM. We will show that when the cavity field and the atom are in exact resonance, the initial field components will split into superposition of four NCSs $\pi/2$ out of phase at the collapses of atomic inversion.

The f-deformed JCM Hamiltonian can be written in terms of the generalized annihilation and creation operators (35, 36):

$$\hat{\mathcal{H}} = \hbar\omega_0\hat{A}^\dagger\hat{A} + \frac{1}{2}\hbar\omega\hat{\sigma}_3 + \hbar\lambda(\hat{\sigma}_+\hat{A} + \hat{A}^\dagger\hat{\sigma}_-), \tag{40}$$

where $\hat{\sigma}_3 = |e\rangle\langle e| - |g\rangle\langle g|$, $\hat{\sigma}_+ = |e\rangle\langle g|$ and $\hat{\sigma}_- = |g\rangle\langle e|$ are atomic operators with commutation relations $[\hat{\sigma}_3, \hat{\sigma}_\pm] = \pm 2\hat{\sigma}_\pm$ and $[\hat{\sigma}_+, \hat{\sigma}_-] = \hat{\sigma}_3$. The excited and the ground states of the atom are denoted by $|e\rangle$ and $|g\rangle$, respectively. Also ω_0 and ω are the field and the atomic transition frequencies and λ is the constant atom-field coupling. The first term in Equation (40) is the Hamiltonian of the deformed free field and may be considered as the 'normally-ordered' Man'ko et al Hamiltonian (38, 39). The second term is the energy operator of the atom. The interaction part of the Hamiltonian is given by the third term in which the atom-field coupling is intensity-dependent due to the existence of the nonlinearity function $f(\hat{n})$. It is clear that the main difference between (40) and standard JCM Hamiltonian is a result of the nonlinearity nature of $f(\hat{n})$. This means that the physical properties of the interaction system described by (40) can be controlled by the nonlinearity function and consequently the appropriate choice of $f(\hat{n})$ may lead to special nonlinearity of the atom-field interaction.

Choosing $j = 2$ in (10) would develop some important physical consequences. First of all, the f-deformed Hamiltonian (40) describes a system in which an atom is surrounded by a Kerr-like medium. Furthermore, the existence of such nonlinearity function in the interaction

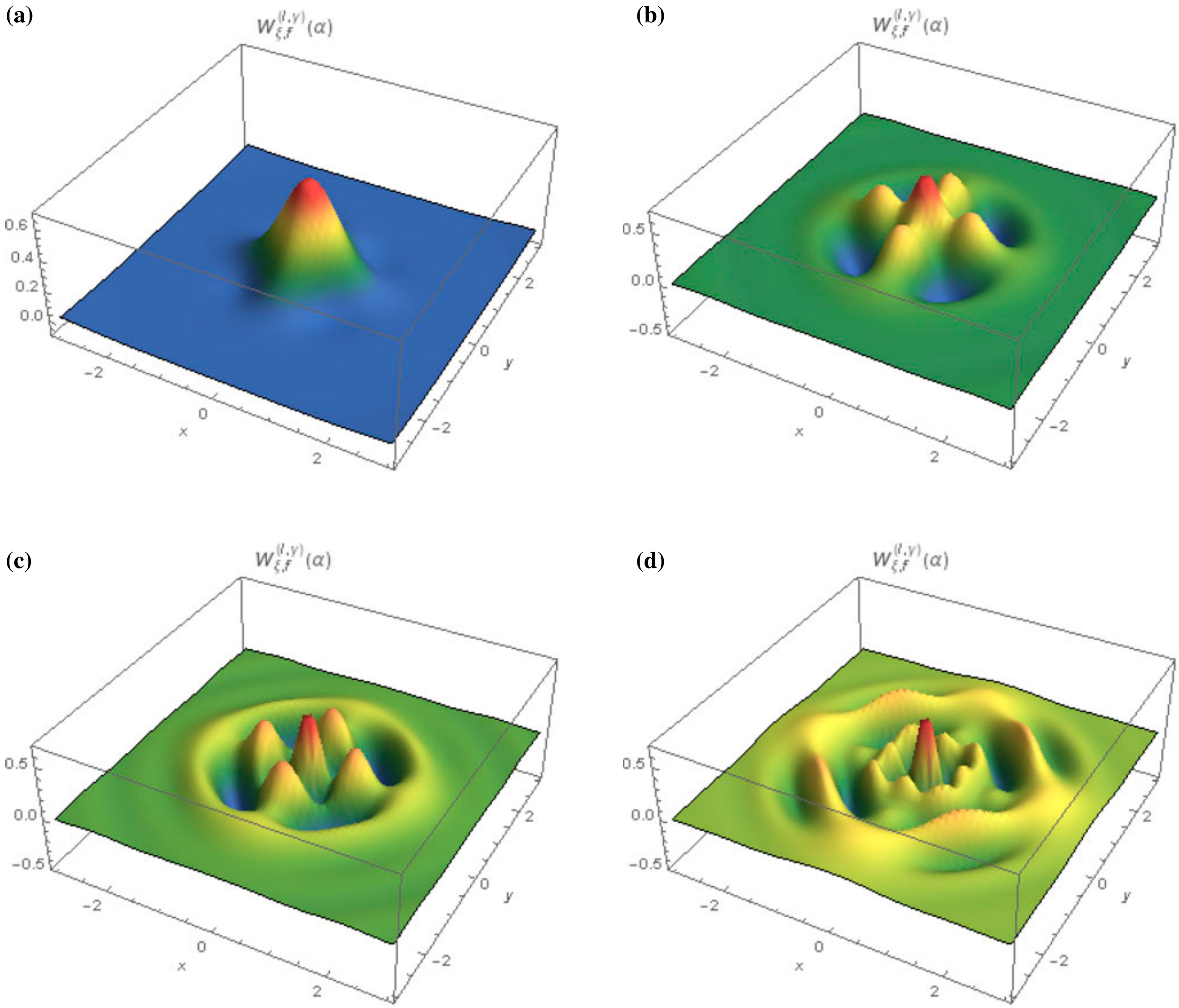

Figure 4. The same as in Figure 3 but for the Wigner distribution function $W^{(l,\gamma)}_{\xi,f}(\alpha)$.

part of the Hamiltonian results in a special intensity-dependent coupling because we deal with $\lambda f(\hat{n})$ instead of the atom-field constant coupling λ of the standard JCM. Using the nonlinearity function (10) to construct the f-deformed annihilation and creation operators makes it possible to use the field f-deformed CSs (NCSs) and their evolutions.

The density operator of the system under study follows a unitary time evolution generated by the time evolution operator $U(t)$. The time evolved density operator can be expressed as $\hat{\rho}(t) = \hat{U}(t)\hat{\rho}(0)\hat{U}^{\dagger}(t)$ where $\hat{\rho}(t)$ is the density operator of the combined atom-field system. The atom is assumed to be initial in its excited state and that the atom and the field be uncorrelated at $t = 0$. Thus, the initial density operator in the atomic basis can be written as (33):

$$\hat{\rho}(0) = \hat{\rho}_A(0) \otimes \hat{\rho}_F(0) = \begin{pmatrix} \hat{\rho}_F(0) & 0 \\ 0 & 0 \end{pmatrix}, \qquad (41)$$

where $\hat{\rho}_{A(F)}$ are the reduced density operators and the subscript $A(F)$ denotes the atomic (field) subsystem. Now considering the Hamiltonian (40) and the nonlinearity function (10) with $j = 2$, the time-dependent density operator for the composed system is found to be (36):

$$\hat{\rho}(t) = \begin{pmatrix} \hat{\mathcal{R}}\hat{\rho}_F(0)\hat{\mathcal{R}}^{\dagger} & \hat{\mathcal{R}}\hat{\rho}_F(0)\hat{\mathcal{S}}^{\dagger} \\ \hat{\mathcal{S}}\hat{\rho}_F(0)\hat{\mathcal{R}}^{\dagger} & \hat{\mathcal{S}}\hat{\rho}_F(0)\hat{\mathcal{S}}^{\dagger} \end{pmatrix}, \qquad (42)$$

where:

$$\hat{\mathcal{R}} = \cos\hat{\Phi}_n t - i\frac{\hat{\Lambda}_n}{2\hat{\Phi}_n}\sin\hat{\Phi}_n t, \quad \hat{\mathcal{S}} = -i\lambda\hat{A}^{\dagger}\frac{\sin\hat{\Phi}_n t}{\hat{\Phi}_n}. \qquad (43)$$

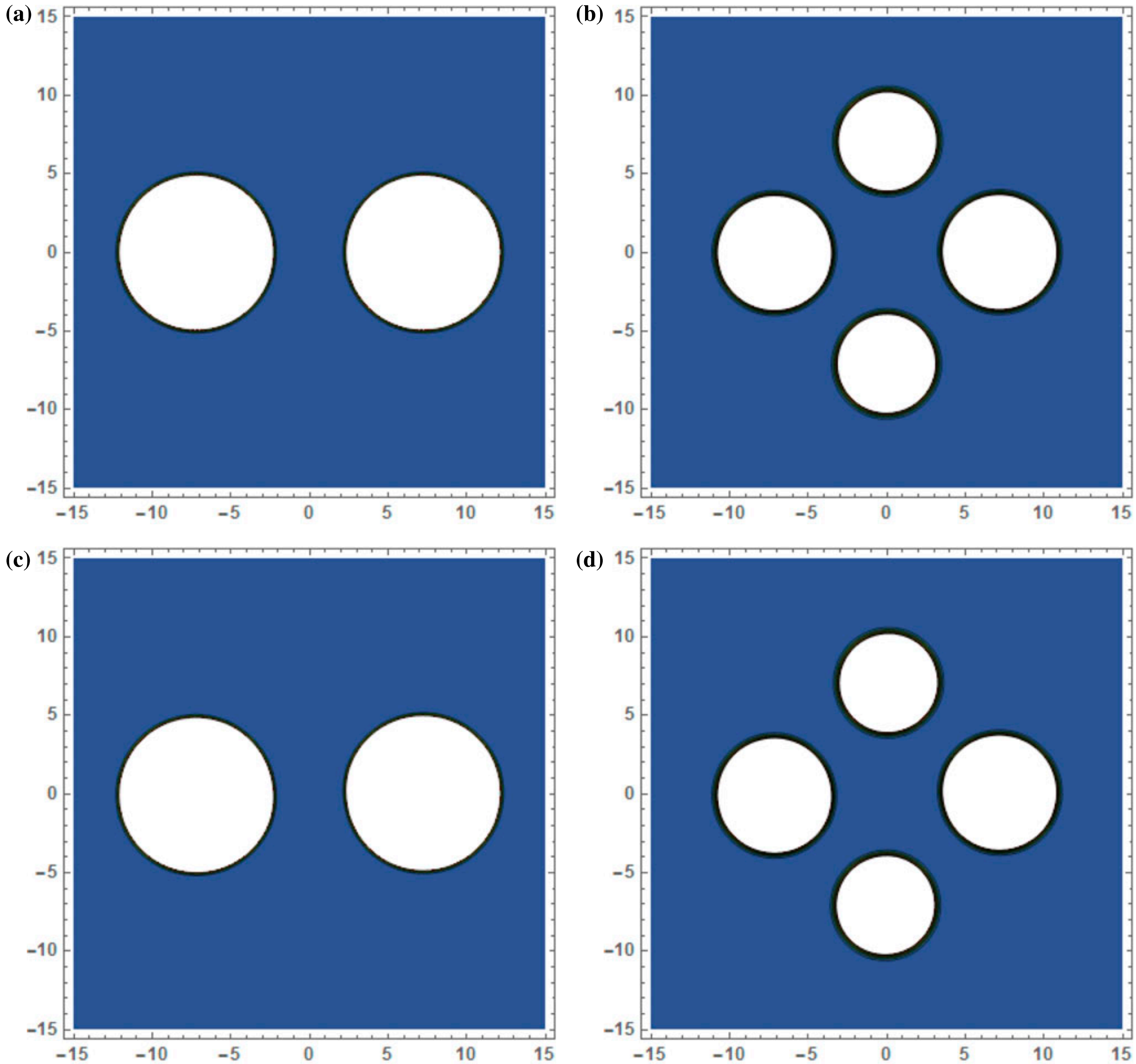

Figure 5. Evolution of the generalized Husimi Q-function for the cavity field in f-deformed JCM when initial field is an ENCS and $\Delta = 0$ for (a) $T = 0$ (b) $T = T_r/2$ (c) $T = T_r$ and (d) $T = 3T_r/2$. The horizontal axis is $\mathrm{Re}(\beta)$ and the Vertical axis is $\mathrm{Im}(\beta)$. The parameters are $\chi/\omega_0 = \lambda/3, \lambda = 0.003, |\alpha_0|^2 = 50$.

In the above expression, the generalized Rabi frequency $\hat{\Phi}_n$ is:

$$\hat{\Phi}_n = \sqrt{\lambda^2(\hat{n}+1)f^2(\hat{n}+1) + \hat{\Lambda}_n^2/4}, \qquad (44)$$

and $\hat{\Lambda}_n = \omega - \omega_0[\hat{A}, \hat{A}^\dagger]$ is the generalized detuning. Also $\omega_0[\hat{A}, \hat{A}^\dagger] \equiv \omega_0(n)$ can be interpreted as an intensity-dependent frequency and we have $[\hat{A}, \hat{A}^\dagger] = 1 + (1/2\gamma)\hat{n}$. Now we assume that the initial single-mode of the electromagnetic field state inside the cavity is an ENCS (that is given by (1) with '+') for which the density operator can be constructed as:

$$\hat{\rho}_F(0) = |N_+|^2 \Big(|\alpha, f\rangle\langle\alpha, f| + |\alpha, f\rangle\langle -\alpha, f| + |-\alpha, f\rangle\langle\alpha, f| + |-\alpha, f\rangle\langle -\alpha, f| \Big). \qquad (45)$$

Replacing (45), as the initial field density operator, in (42) and using (43), we can find the explicit form of the atom-field system density operator $\hat{\rho}(t)$. The photon number distribution of the field at time t is defined as $P_n(t) = \langle n|\hat{\rho}_F(t)|n\rangle$. On the other hand, the time evolved density operator of the cavity field can be easily obtained by tracing the compound system density operator over the atomic variables in (42), as follows:

$$\hat{\rho}_F(t) = Tr_A[\hat{\rho}(t)] = \hat{\mathcal{R}}\hat{\rho}_F(0)\hat{\mathcal{R}}^\dagger + \hat{\mathcal{S}}\hat{\rho}_F(0)\hat{\mathcal{S}}^\dagger. \quad (46)$$

Regarding the dynamics of the system which is very susceptible to the photon statistics of the initial cavity field, the photon number distribution of the field at $t = 0$ (the probability of finding n photon in the initial field) is given by:

$$P_{2n}(0) = \frac{2|\alpha_0|^{2n}(4\gamma)^n}{n!\,\mathcal{F}(\gamma, |\alpha_0|^2)(4\gamma)_n}, \quad (47)$$

with $\mathcal{F}(\gamma, |\alpha_0|^2) = {}_0F_1(4\gamma; 4\gamma|\alpha_0|^2) + {}_0F_1(4\gamma; -4\gamma|\alpha_0|^2)$ and we have $P_{2n+1}(0) = 0$. In Equation (47), $|\alpha_0|^2$ is the initial mean photon number of the field CS and we use (45) as the initial field density operator $\hat{\rho}_F(0)$.

The evolution of the field Q-function is directly connected to the collapses and revivals evolution of the atomic inversion (*67*). In order to explain the pattern of collapses and revivals of the generalized Rabi oscillations given by Equation (44), we recall that the atomic inversion is defined as the expectation value of the atomic transition operator $\langle\hat{\sigma}_3\rangle = Tr[\hat{\rho}(t)\hat{\sigma}_3]$. Using (42) and (43), the explicit form of the atomic inversion is obtained:

$$\langle\hat{\sigma}_3\rangle = \sum_{n=0}^{\infty} P_n(0)\left(\frac{\hat{\Lambda}_n^2}{4\hat{\Phi}_n^2} + \frac{\lambda^2(n+1)f^2(n+1)}{\hat{\Phi}_n^2}\cos(2\hat{\Phi}_n t)\right). \quad (48)$$

Now considering that the collapses occur when the various terms in the summation (48) become uncorrelated and knowing that for an ENCS as the initial cavity field, $P_n(0)$ is zero for odd 'n' , we can calculate the revival time (the time in which the revivals will occur) by estimating the time that the two non-zero neighbour terms will be in phase again:

$$T_r(2\hat{\Phi}_{\bar{n}+2} - 2\hat{\Phi}_{\bar{n}}) \approx 2\pi, \quad (49)$$

where $\hat{\Phi}_n$ is given by Equation (44) and in (49), we put $n \approx \bar{n}$. The generalized form of the Husimi Q-function for the f-deformed JCM has the explicit form $Q_f(\beta) = 1/\pi\langle\beta,f|\hat{\rho}_F(t)|\beta,f\rangle$. The contour plots of the cavity field Husimi Q-function at some specific times $T = \lambda t$ are illustrated in Figure 5. The parameters are $\lambda = 0.003$, $\chi/\omega_0 = \lambda/3$ and the initial field intensity fixed at $|\alpha_0|^2 = 50 \approx \bar{n}$. Also we assume that the atom and the cavity field is in exact resonance i.e. $\Delta = 0$.

Considering that the field is initially at an ENCS, we clearly have two peaks at $T = 0$ corresponding to each constituent (see Figure 5(a)). Due to the collapse of Rabi oscillations at $T = T_r/2$ (the first collapse of atomic inversion), the two starting peaks break into four well-localized components centred at $x = y = 0$ in the phase space (Figure 5(b)). The four separated components do not hold the same probabilities because of the intensity-dependent coupling between the atom and the field. However, it is evident that these four peaks correspond to the interference of the four-component NCSs which are exactly $\pi/2$ out of phase. Indeed in the interaction of a single two-level atom with an ENCS, in the full nonlinear regime and the exact resonance condition, we start with a f-deformed Schrödinger cat state which results from the interference of the states $|\alpha,f\rangle$ and $|-\alpha,f\rangle$. Then due to the interaction between the atom and the field, we deal with four NCSs at the collapse time which arise from the interference of $|\pm\alpha,f\rangle$ and $|\pm i\alpha,f\rangle$. Such a superposition of NCSs can be interpreted as the deformed Schrödinger cat-like states but with combination of four macroscopic states instead of two. This situation is completely similar to the case of FPNCSs that is expressed for $l = 0$ in Equation (9) except that the constituent states of the superposition in the interacting system, do not hold the same weight. Nevertheless, considering the similarity of the Husimi Q-function of FPNCSs to the four peak structure of the field in our interacting system, one can conclude that, in exact resonance condition, the field at the collapse time of the atomic inversion can be a possible source for such states. In other words, the initial cavity field is the eigenstate of $\hat{A}^2$ while, at the collapse time, it might transform in to an eigenstate of $\hat{A}^4$. Due to the intricate evolution of the field density operator, the photon number distribution of the cavity field evolves in a complicated way. Therefore, the proof of this theory needs more detailed discussion. Furthermore, recombination and splitting of the field components for a successive revival ($T = T_r$) and collapse ($T = 3T_r/2$) are respectively clear in Figure 5(c) and 5(d). Finally, it is worth mentioning that the same behaviour comes along for the linear system accessible by setting $f(n) = 1$ or equivalently $\chi = 0$ in our model (*29*).

7. Conclusions

We have introduced the FPNCSs as the eigenstates of biquadratic of the generalized photon annihilation operator $\hat{A}^4$. A special feature of the presented states is that they can be constructed as superpositions of four NCSs $\pi/2$ out of phase. Actually, such states are deformed 'Schrödinger cat' type states with four components which can play an important role in many quantum information processing tasks covering quantum computation and quantum teleportation. In order to extract the nonlinear effects, in all cases, we perform comparisons of FPNCSs and the corresponding linear states in (*11*) which can be recovered by setting $f(\hat{n}) = 1$ in our formalism. As a natural signature, the study of the photon count probability function shows a strong oscillatory behaviour which illustrates the nonclassicality of the introduced states. The domain of nonclassical features such as sub-Poissonian statistics and squeezing effects of FPNCSs is rather extensive in comparison to the corresponding features in FPSCSs. We should emphasize, unlike FPSCSs, sub-Poissonian behaviour is to be considered a characteristic of FPNCSs in a vast region of $|\xi|$. Also the study of the higher order squeezing as well as the amplitude-squared squeezing reflects the fact that the domain of squeezing in FPNCSs is larger than that for the corresponding standard states. As a prominent feature of the states (6), it can conveniently be understood that the depth or domain of the nonclassicality features is adjustable by changing the magnitude of nonlinearity parameters.

The Glauber–Sudarshan *P*-function of FPNCSs is found to be a highly singular function which is an indication of the high degree of nonclassicality of these states. The importance of this issue is that the necessary and sufficient condition for nonclassicality of an arbitrary quantum state that has been presented by T. Kiesel and W. Vogel is based on the regularized *P*-function which can uncover all the negativities of the original *P*-function. This strategy for extracting nonclassical characteristics of the states (6) should be followed in a separate study. Investigating the other quasi-probability distributions indicates that, as might be expected, the Husimi *Q*-function of the presented states has a four peak structure over higher values of $|\xi|$. Also the negativity of Wigner function has been justified for all sets of FPNCSs, therefore, one can conclude that this type of nonclassicality is a natural property of these states.

Finally, using the solution of f-deformed JCM based on the density operator method, we study the interaction of a single two-level atom with an ENCS in a lossless optical cavity. It is found that when the atom and the cavity field are in exact resonance, the initial two peaks in the phase space would split into four well-localized peaks during successive collapses of atomic inversion. Although the field state at collapse time is not necessarily a FPNCSs, the similarity of the field Husimi *Q*-function in this system to the four peak structure of the presented states in this article, reinforces the idea that the interaction of an ENCS with a single two-level atom in the framework of f-deformed JCM can be considered as a possible source for generating such states.

At the end of this paper, we mention that by setting $f(\hat{n}) = 1$, our presented formalism readily recovers all results of (*11*) as a special case. Furthermore, it is worth noting that this study can be extended to the notion of k-photon NCSs as the eigenstates of the kth power of the generalized annihilation operator. This work is in preparation and will be submitted in the near future.

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