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On the convergence analysis of the Tau method applied to fourth-order partial differential equation based on Volterra-Fredholm integral equations

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ABSTRACT

This paper presents a study of the performance of the Tau method using orthogonal polynomials as basis functions for solving fourth-order partial differential equations with boundary conditions. Because of the good numerical stability properties of integral operators in compare to differential operator, we first convert this PDE problem to Volterra-Fredholm integral equation and then apply the numerical Tau method to solve the obtained integral equation. Applying the Tau method yields a system of the ordinary differential equation such that this system is solved by piecewise polynomial collocation methods. Convergence analysis and error estimation of the Tau method are discussed. The advantages of converting PDE to integral equation are shown by the numerical examples. For this aim, we consider two cases to solve the proposed examples. In case1, we apply the Tau method to solve the converted problem (Volterra-Fredholm integral equation) and in case 2, we solve PDE problem directly by Tau method. Comparing the numerical results, we observe that the obtained errors in case 1 are less than the errors in case 2.

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1. Introduction

The Tau method was introduced by Lannczos in his memoir of 1938 [20] to introduce the use of Chebyshev polynomials concerning the solution of linear differential equations with polynomial coefficients, in terms of finite expansions. The perturbation term is expressed in terms of polynomials with free coefficients (the Tau parameters in Ortiz [28]). A norm of these polynomials satisfies a minimum condition; Chebyshev polynomials are often a convenient choice, but other types of polynomials have been used in practice. For a discussion of Chebyshev, Legendre and other types of perturbation terms and their influence on the error of approximation of the Tau method see Namasivayam and Ortiz [27]. The approach of [29], called operational by its authors, leads to the coefficients of a Tau bivariate approximate solution of a linear partial differential equation provided its coefficients and the right-hand side of the equation are polynomials (or rational polynomials) or polynomials approximation of a given accuracy of the original functions. Since then, this method has been extended in different ways for solving variant kinds of differential and partial differential equations and also used in scientific computational which are resulted from linear differential equations involved with polynomial coefficients (see [22]). We shall not discuss here the details of the construction of the Tau approximate solution of such problem; the reader can find a detailed discussion in the paper of Ortiz and Samara [29].

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The Tau method has found extensive application in recent years presented in different kinds of numerical solutions of ordinary differential equations (ODEs) (see [10,11,13–16,29]) and in for the case of numerical solution of partial differential equations (PDEs) (see [9,30,16,18,25,27]). In some papers have been discussed the application of the Chebyshev and Legendre polynomials and their numerical merits in solving ODEs and PDEs numerically (for example, [7,35,10,11,13–18,20,21,25,29,21]). In the literature of numerical analysis, there are many fourth-order linear and non-linear boundary value problems arising in science and engineering which are solved either analytically or numerically [6,8,19,33,34]. For this, many authors have attempted to solve the fourth-order boundary value problem (BVP) to obtain high accuracy rapidly by using numerous methods, such as the least square method, finite difference method, Sinc-Galerkin method, and also some other methods using polynomial and non-polynomial spline functions [12,31,32].

We are, therefore, motivated to work in this direction of extending the Tau method with arbitrary bases for the solution of the fourth-order boundary value PDE problem in the following form:

$$\frac{\partial \phi}{\partial t} = \beta_1 \frac{\partial^2 \phi}{\partial x^2} + \beta_2 \frac{\partial^4 \phi}{\partial x^4}, \quad x \in I = [a,b],\ t > 0, \tag{1}$$

with Dirichlet boundary conditions

$$\phi(x,0) = f(x),\ \ \phi(a,t) = g_1(t),\ \ \phi(b,t) = g_2(t),$$

and with Neumann boundry conditions

$$\frac{\partial^2 \phi(a,t)}{\partial x^2} = h_1(t),\ \ \frac{\partial^2 \phi(b,t)}{\partial x^2} = h_2(t).$$

In (1), by considering $\beta_1 = k$ (k is the thermal conductivity) and $\beta_2 = \frac{k\Delta^2 x}{2}\big(\frac{(m+1)}{6} - \alpha(2m+1)\big)$ with $\alpha = \frac{k\Delta t}{\Delta^2 x}$ where m is an integer and $m\Delta t$ denotes the upper bound on the delay due to communication in the small interval $(x, x+\Delta x)$, we encounter the following fourth-order boundary value PDE problem

$$\frac{\partial \phi}{\partial t} = k\frac{\partial^2 \phi}{\partial x^2} + \frac{k\Delta^2 x}{2}\frac{\partial^4 \phi}{\partial x^4}\big(\frac{(m+1)}{6} - \alpha(2m+1)\big),$$

which appears in physics with the important applications [2,3,26].

Here, we convert this PDE problem to Volterra-Fredholm integral equation and then apply the numerical Tau method to solve the obtained integral equation.

The outline of the paper is as follows. First, in Section 2, we investigate the structure of the numerical solution of the equation (1) by transforming it into the Volterra-Fredholm integral equation and the spectral Tau method based on orthogonal polynomials is applied for the numerical solution of the fourth-order boundary value problem (1). In Section 3, we analyze the convergence of numerical solution which is obtained by Tau method. Some numerical examples are presented in Section 4, where the exact solutions are explicitly given. Finally, in section 5, the conclusion of this work will be presented.

2. Polynomial spectral Tau method

In this section, we first convert PDE problem (1) to Volterra-Fredholm integral equation. Because of the good numerical stability properties of integral operators in comparison to the differential operator, this converting sometimes may be useful from a numerical point of view. This advantage will be shown by the numerical examples in section 4. Let consider rewrite (1) as

$$\frac{\partial^4 \phi}{\partial x^4} = \frac{1}{\beta_2}\frac{\partial \phi}{\partial t} - \frac{\beta_1}{\beta_2}\frac{\partial^2 \phi}{\partial x^2}, \quad x \in I = [a,b],\ t > 0. \tag{2}$$

We aim is to transform the BVP (2) into a linear Volterra-Fredholm equation. Therefore, we assume that $\frac{\partial^4 \phi(x,t)}{\partial x^4} = u(x,t)$. By integrating both sides of the mentioned relation in the interval $[a,x]$, we have

$$\frac{\partial^3 \phi(x,t)}{\partial x^3} = \frac{\partial^3 \phi(a,t)}{\partial x^3} + \int_a^x u(t_1,t)dt_1. \tag{3}$$

Now, we integrate both sides of the above equation in the interval $[a,x]$ as

$$\begin{aligned}\frac{\partial^2 \phi(x,t)}{\partial x^2} &= \frac{\partial^2 \phi(a,t)}{\partial x^2} + (x-a)\frac{\partial^3 \phi(a,t)}{\partial x^3} + \int_a^x \int_a^{t_1} u(t_2,t)dt_2 dt_1 \\ &= \frac{\partial^2 \phi(a,t)}{\partial x^2} + (x-a)\frac{\partial^3 \phi(a,t)}{\partial x^3} + \int_a^x (x-t_1)u(t_1,t)dt_1.\end{aligned} \tag{4}$$

We consider $x=b$ in (4), then is obtained as follows

$$\frac{\partial^2\phi(b,t)}{\partial x^2}=\frac{\partial^2\phi(a,t)}{\partial x^2}+(b-a)\frac{\partial^3\phi(a,t)}{\partial x^3}+\int_a^b(b-t_1)u(t_1,t)dt_1,$$

from boundary conditions, it can be gained as following

$$\frac{\partial^3\phi(a,t)}{\partial x^3}=\frac{1}{b-a}\big(h_2(t)-h_1(t)\big)-\frac{1}{b-a}\int_a^b(b-t_1)u(t_1,t)dt_1. \tag{5}$$

By integrating from (4) in interval $[a,x]$, it's obtained

$$\begin{aligned}\frac{\partial\phi(x,t)}{\partial x}&=\frac{\partial\phi(a,t)}{\partial x}+h_1(t)(x-a)+\frac{1}{2}(x-a)^2\frac{\partial^3\phi(a,t)}{\partial x^3}\\&\quad+\int_a^x\int_a^{t_1}(x-t_2)u(t_2,t)dt_2dt_1\\&=\frac{\partial\phi(a,t)}{\partial x}+h_1(t)(x-a)+\frac{1}{2}(x-a)^2\frac{\partial^3\phi(a,t)}{\partial x^3}\\&\quad+\int_a^x\frac{1}{2}(x-t_1)^2u(t_1,t)dt_1.\end{aligned} \tag{6}$$

Now by integrating both sides of (6), we have

$$\begin{aligned}\phi(x,t)&=\phi(a,t)+(x-a)\frac{\partial\phi(a,t)}{\partial x}+\frac{1}{2}h_1(t)(x-a)^2\\&\quad+\frac{1}{6}(x-a)^3\frac{\partial^3\phi(a,t)}{\partial x^3}+\int_a^x\frac{1}{6}(x-t_1)^3u(t_1,t)dt_1.\end{aligned} \tag{7}$$

By considering $x=b$ in (7) and using boundary conditions, we can obtain

$$\begin{aligned}\frac{\partial\phi(a,t)}{\partial x}&=\frac{1}{(b-a)}\big(g_2(t)-g_1(t)\big)-\frac{1}{2}(b-a)h_1(t)\\&\quad-\frac{1}{6}(b-a)^2\frac{\partial^3\phi(a,t)}{\partial x^3}-\frac{1}{6(b-a)}\int_a^b(b-t_1)^3u(t_1,t)dt_1.\end{aligned} \tag{8}$$

Now by replacing $\frac{\partial^3\phi(a,t)}{\partial x^3}$ from (5) in (4), we have

$$\begin{aligned}\frac{\partial^2\phi(x,t)}{\partial x^2}&=h_1(t)+\frac{(x-a)}{(b-a)}\big(h_2(t)-h_1(t)\big)\\&\quad-\frac{(x-a)}{(b-a)}\int_a^b(b-t_1)u(t_1,t)dt_1+\int_a^x(x-t_1)u(t_1,t)dt_1,\end{aligned} \tag{9}$$

also by replacing $\frac{\partial^3\phi(a,t)}{\partial x^3}$ from (5) and $\frac{\partial\phi(a,t)}{\partial x}$ from (8) in (7), we conclude

$$
\begin{aligned}
\phi(x,t) &= g_1(t) + \frac{(x-a)}{(b-a)}\big(g_2(t)-g_1(t)\big) + \frac{1}{2}(x-a)(x-b)h_1(t) \\
&\quad + \frac{(x-a)}{6(b-a)}(x^2-2ax+2ab-b^2)(h_2(t)-h_1(t)) \\
&\quad + \int_a^b \frac{(a-x)}{6(b-a)}\Big((x^2-2ax+2ab-b^2)(b-t_1)+(b-t_1)^3\Big)u(t_1,t)dt_1 \\
&\quad + \int_a^x \frac{1}{6}(x-t_1)^3 u(t_1,t)dt_1.
\end{aligned} \tag{10}
$$

Now by derivative from two hand side of (10) respect to t it is deduced that

$$
\begin{aligned}
\frac{\partial\phi(x,t)}{\partial t} &= g'_1(t) + \frac{(x-a)}{(b-a)}\big(g'_2(t)-g'_1(t)\big) + \frac{1}{2}(x-a)(x-b)h'_1(t) \\
&\quad + \frac{(x-a)}{6(b-a)}(x^2-2ax+2ab-b^2)(h'_2(t)-h'_1(t)) \\
&\quad + \int_a^b \frac{(a-x)}{6(b-a)}\Big((x^2-2ax+2ab-b^2)(b-t_1)+(b-t_1)^3\Big)\frac{\partial u(t_1,t)}{\partial t}dt_1 \\
&\quad + \int_a^x \frac{1}{6}(x-t_1)^3 \frac{\partial u(t_1,t)}{\partial t}dt_1.
\end{aligned} \tag{11}
$$

Finally, inserting the equations (9) and (11) into (2), lead to the following linear Volterra-Fredholm integral equation

$$
\begin{aligned}
u(x,t) &= \hat{f}(x,t) + \int_a^b K_1(x,t_1)u(t_1,t)dt_1 + \int_a^b K_2(x,t_1)\frac{\partial u(t_1,t)}{\partial t}dt_1 \\
&\quad + \int_a^x K_3(x,t_1)u(t_1,t)dt_1 + \int_a^x K_4(x,t_1)\frac{\partial u(t_1,t)}{\partial t}dt_1,
\end{aligned} \tag{12}
$$

where

$$
\begin{aligned}
\hat{f}(x,t) &= \frac{1}{\beta_2}g'_1(t) - \frac{\beta_1}{\beta_2}h_1(t) + \frac{(x-a)}{\beta_2(b-a)}(g'_2(t)-g'_1(t)) + \frac{1}{2\beta_2}(x-a)(x-b)h'_1(t) \\
&\quad + \frac{(x-a)}{6\beta_2(b-a)}(x^2-2ax+2ab-b^2)(h'_2(t)-h'_1(t)) - \frac{\beta_1(x-a)}{\beta_2(b-a)}(h_2(t)-h_1(t)), \\
K_1(x,t_1) &= \frac{\beta_1(x-a)}{\beta_2(b-a)}(b-t_1)\ , \\
K_2(x,t_1) &= \frac{(a-x)}{6\beta_2(b-a)}\Big((x^2-2ax+2ab-b^2)(b-t_1)+(b-t_1)^3\Big), \\
K_3(x,t_1) &= -\frac{\beta_1}{\beta_2}(x-t_1)\ , \quad K_4(x,t_1) = \frac{1}{6\beta_2}(x-t_1)^3.
\end{aligned}
$$

Remark 2.1. Note that the existence and uniqueness theorem of the solution of the PDE problem (1) can be investigated by the existence, uniqueness and structure of the solution of the Volterra-Fredholm integral equation (12) from [1].

Now, in the spectral Tau method, we seek an approximate solution of the equation (12) as:

$$
u_N(x,t) = \sum_{k=0}^{N} a_k(t)\psi_k(x), \tag{13}
$$

where $\psi_k(x), x \in [a,b]$ is a set of orthogonal polynomials with respect to the inner product

$$\left\langle \psi_k(x), \psi_j(x) \right\rangle_w = \int_a^b \psi_k(x)\psi_j(x)w(x)dx.$$

Inserting (13) into (12), we obtain

$$\sum_{k=0}^{N} a_k(t)\psi_k(x) = \hat{f}(x,t) + \sum_{k=0}^{N} a_k(t)(\alpha_{1k}(x) + \alpha_{3k}(x)) + a'_k(t)(\alpha_{2k}(x) + \alpha_{4k}(x)), \tag{14}$$

where

$$\alpha_{1k}(x) = \int_a^b K_1(x,t)\psi_k(t)dt, \quad \alpha_{2k}(x) = \int_a^b K_2(x,t)\psi_k(t)dt,$$
$$\alpha_{3k}(x) = \int_a^x K_3(x,t)\psi_k(t)dt, \quad \alpha_{4k}(x) = \int_a^x K_4(x,t)\psi_k(t)dt, \ k = 0, \cdots, N.$$

Projecting (14) on the $\{\psi_l(x)\}_{l=0}^{N}$

$$a_l(t)\left\langle \psi_l(x), \psi_l(x) \right\rangle_w = \left\langle \hat{f}(x,t), \psi_l(x) \right\rangle_w + \sum_{k=0}^{N} a_k(t)\lambda_{kl} + a'_k(t)\eta_{kl}, \ l = 0, \cdots, N, \tag{15}$$

where

$$\lambda_{kl} = \left\langle \alpha_{1k}(x) + \alpha_{3k}(x), \psi_l(x) \right\rangle_w, \ \ \eta_{kl} = \left\langle \alpha_{2k}(x) + \alpha_{4k}(x), \psi_l(x) \right\rangle_w.$$

Let

$$\mathbf{\Upsilon} = \{\lambda_{kl}\}_{k,l=0}^{N}, \mathbf{D} = diag(\left\langle \psi_0(x), \psi_0(x) \right\rangle_w, \cdots, \left\langle \psi_N(x), \psi_N(x) \right\rangle_w), \mathbf{\Lambda} = \{\eta_{kl}\}_{k,l=0}^{N},$$

and

$$\mathbf{F}(t) = (\left\langle \hat{f}(x,t), \psi_0(x) \right\rangle_w, \cdots, \left\langle \hat{f}(x,t), \psi_N(x) \right\rangle_w)^T.$$

This procedure yields an initial-value problem of $(N+1)$ equations with $N+1$ unknowns, $\bar{\mathbf{a}} = (a_0, \dots, a_N)^T$, of the form

$$\mathbf{\Lambda}\bar{\mathbf{a}}'(t) = (\mathbf{D} - \mathbf{\Upsilon})\bar{\mathbf{a}}(t) - \mathbf{F}(t), \tag{16}$$

with the initial conditions

$$a_l(0) = \frac{\left\langle \frac{d^4 f(x)}{dx^4}, \psi_l(x) \right\rangle_w}{\left\langle \psi_l(x), \psi_l(x) \right\rangle_w}, \quad l = 0, \cdots, N.$$

Remark 2.2. Considering the matrix $\mathbf{\Lambda}$ in the initial-value problem (16), we have two classification:
1) If $det(\mathbf{\Lambda}) \neq 0$, then we have ordinary differential equations (ODEs).
2) If $det(\mathbf{\Lambda}) = 0$, then we have differential-algebraic equations (DAEs).

Now, we apply piecewise polynomial collocation method [4] for the approximate solution of the initial-value problem (16) as:

Let $I_h = \{0 = t_0 < t_1 < ... < t_n = T\}$ be a uniform partition of the interval $[0, T]$ with grid points $t_n = nh$ $(n = 0, ..., N_1)$. Also let h be the stepsize and Θ is given by:

$$\Theta = \{t_{nj} = t_n + c_j h : \ 0 \le c_1 < c_2 < ... < c_m \le 1, \qquad 0 \le n \le N_1 - 1\},$$

where c_j $(j = 1, \cdots, m)$ and t_{nj} are the collocation parameters and the collocation points, respectively. We define the subintervals $\sigma_n = (t_n, t_{n+1}], n = 0, \dots, N_1 - 1$, and the space of piecewise polynomials of degree $m \ge 0$, as follows:

$$S_m^0(I_h) = \left\{\upsilon \in C(I) : \upsilon|_{\sigma_n} \in \Pi_m, \quad (0 \le n \le N_1 - 1)\right\},$$

where Π_m denotes the space of all real polynomials of degree not exceeding m. The collocation solution $\bar{\mathbf{a}}_h \in S_m^0(I_h)$ for (16) is defined by the collocation equation

$$\mathbf{\Lambda}\bar{\mathbf{a}}_h'(t) = (\mathbf{D} - \mathbf{\Upsilon})\bar{\mathbf{a}}_h(t) - \mathbf{F}(t), \quad t \in \Theta, \quad \bar{\mathbf{a}}_h(0) = \bar{\mathbf{a}}(0) = \bar{\mathbf{a}}_0. \tag{17}$$

Since $\bar{\mathbf{a}}_h'|_{\sigma_n} \in \Pi_{m-1}$, for $\rho \in [0, 1]$, the following relations hold:

$$\bar{\mathbf{a}}_h'(t_n + \rho h) = \sum_{j=1}^{m} L_j(\rho)\bar{\mathbf{A}}_{nj}, \qquad \bar{\mathbf{A}}_{nj} = \bar{\mathbf{a}}_h'(t_n + c_j h), \tag{18}$$

where $L_j(\rho)$ represents the Lagrange canonical polynomials for the collocation parameters $\{c_j\}$ and is defined as:

$$L_j(\rho) = \prod_{k \neq j} \frac{(\rho - c_k)}{(c_j - c_k)}, \qquad j = 1, ..., m.$$

Setting $\bar{\mathbf{a}}_n = \bar{\mathbf{a}}_h(t_n)$ and $\beta_j(\rho) = \int_0^{\rho} L_j(s)ds, \quad j = 1, \ldots, m$, for $\bar{\mathbf{a}}_h \in S_m^0(I_h)$ we have

$$\bar{\mathbf{a}}_h(t_n + \rho h) = \bar{\mathbf{a}}_n + h\sum_{j=1}^{m} \beta_j(\rho)\bar{\mathbf{A}}_{nj}, \qquad \rho \in [0, 1]. \tag{19}$$

Inserting (18) and (19) into collocation equation (17), we obtain

$$\mathbf{\Lambda}\bar{\mathbf{A}}_{ni} = (\mathbf{D} - \mathbf{\Upsilon})(\bar{\mathbf{a}}_n + h\sum_{j=1}^{m} \beta_j(c_i)\bar{\mathbf{A}}_{nj}) - \mathbf{F}(t_{ni}) \;\; i = 1, \ldots, m. \tag{20}$$

Now, by substituting $\bar{\mathbf{A}}_{nj}$ as the solution of the system of linear algebraic equations (20) into (19), we can get the approximate solution of the initial-value problem (16) for arbitrary $\rho \in [0, 1]$.

Remark 2.3. Note that the equations (19) and (20) define a continuous implicit Runge–Kutta (CIRK) method [4] for the initial-value problem (16). Also, with assuming $\rho = 1$ in (19), we have classical m-stage implicit Runge–Kutta method for (16) as:

$$\bar{\mathbf{a}}_{n+1} = \bar{\mathbf{a}}_h(t_n + h) = \bar{\mathbf{a}}_n + h\sum_{j=1}^{m} \beta_j(1)\bar{\mathbf{a}}_{nj},$$

with the stage equations (20).

3. Convergence analysis

Due to the use of the shifted Legendre polynomials in the interval $[0, 1]$ as orthogonal basis functions, then in this section, we assume without loss of generality that $I = [0, 1]$. Now we consider the equation (2) in the following form.

$$\begin{cases} \frac{\partial^4\phi}{\partial x^4} = \frac{1}{\beta_2}\frac{\partial\phi}{\partial t} - \frac{\beta_1}{\beta_2}\frac{\partial^2\phi}{\partial x^2}, \\ \phi(x, 0) = f(x), \;\; \phi(0, t) = g_1(t), \;\; \phi(1, t) = g_2(t), \\ \frac{\partial^2\phi(0,t)}{\partial x^2} = h_1(t), \;\; \frac{\partial^2\phi(1,t)}{\partial x^2} = h_2(t). \end{cases} \tag{21}$$

Using a similar procedure as outlined in section 2, we have

$$\begin{aligned} \phi(x, t) = {} & g_1(t) + x(g_2(t) - g_1(t)) + \frac{1}{2}(x^2 - x)h_1(t) + \frac{1}{6}(x^3 - x)(h_2(t) - h_1(t)) \\ & + \int_0^1 -\frac{1}{6}x\Big((x^2 - 1)(1 - t_1) + (1 - t_1)^3\Big)u(t_1, t)dt_1 \\ & + \int_0^x \frac{1}{6}(x - t_1)^3 u(t_1, t)dt_1, \end{aligned} \tag{22}$$

and

$$
\begin{aligned}
u(x,t) = f(x,t) &+ \int_0^1 K_1(x,t_1)u(t_1,t)dt_1 + \int_0^1 K_2(x,t_1)\frac{\partial u(t_1,t)}{\partial t}dt_1 \\
&+ \int_0^x K_3(x,t_1)u(t_1,t)dt_1 + \int_0^x K_4(x,t_1)\frac{\partial u(t_1,t)}{\partial t}dt_1.
\end{aligned} \tag{23}
$$

To prove the error estimate, we need the following lemmas.

Lemma 3.1. *[5] Assume that $H_w^m(\Lambda)$ denotes the Sobolev space of all functions $\phi(\mathbf{x})(\mathbf{x} = (x_1 \cdots, x_p))$ on $\Lambda = (0,1)^p (p = 1,2)$ such that $\phi(\mathbf{x})$ and all its weak derivatives up to order m are in $L_w^2(\Lambda)$. Let $\mathbb{P}_N(\Lambda)$ be the space of all polynomials with a degree not exceeding N on Λ. Denote by P_N the orthogonal projective operator from $L_w^2(\Lambda)$ on to $\mathbb{P}_N(\Lambda)$. For all $\phi \in H_w^m(\Lambda), m \geq 1$, the following estimate hold for the truncation error of shifted Chebyshev or Legendre series*

$$
\|\phi - P_N\phi\|_{L_w^2(\Lambda)} \leq CN^{-m}|\phi|_{H_w^{m,N}(\Lambda)}, \tag{24}
$$

where the semi-norm $|.|$ is defined as

$$
|\phi|_{H_w^{m,N}(\Lambda)} = \left(\sum_{j=min(m,N+1)}^{m} \sum_{i=1}^{p} ||D_i^j \phi||^2_{L_w^2(\Lambda)}\right)^{1/2},
$$

such that $\alpha = (\alpha_1, \cdots, \alpha_p)$ is a nonnegative multi-index with $D^\alpha \phi = \frac{\partial^{\alpha_1+\cdots+\alpha_p}\phi}{\partial x_1^{\alpha_1}\cdots\partial x_p^{\alpha_p}}$.

To prove the error estimate in weighted L_w^2 norm, we need the generalized Hardy's inequality as

Lemma 3.2. *[23] For all measurable function $f \geq 0$, the generalized Hardy's inequality*

$$
\Big(\int_a^b |(Tf)(x)|^q w_1(x)dx\Big)^{\frac{1}{q}} \leq C\Big(\int_a^b |f(x)|^p w_2(x)dx\Big)^{\frac{1}{p}},
$$

holds if and only if

$$
\sup_{a<x<b} \Big(\int_x^b w_1(t)dt\Big)^{\frac{1}{q}} \Big(\int_a^x w_2^{1-p'}(t)dt\Big)^{\frac{1}{p'}} < \infty, \quad p' = \frac{p}{p-1},
$$

for the case $1 < p \leq q < \infty$. Here, T is an operator of the form

$$
(Tf)(x) = \int_a^x k(x,t)f(t)dt,
$$

with $k(x,t)$ a given kernel, w_1 and w_2 weight functions, and $-\infty \leq a < b \leq \infty$.

Now, we consider Gronwall's inequality as

Theorem 3.3. *[24]Let nonnegative function $u(t)$ defined on $[t,\infty)$ satisfies in the following inequality*

$$
u(t) \leq c + \int_{t_0}^t k(t,s)u(s)ds + \int_{t_0}^t \int_{t_0}^s G(t,s,\sigma)u(\sigma)d\sigma ds,
$$

where $k(t,s)$ and $G(t,s,\sigma)$, as $t \geq s \geq \sigma \geq t$ and $c > 0$, are nonnegative continuous derivative functions, then

$$
u(t) \leq c\,exp\{\int_{t_0}^t \Big[k(s,s) + \int_{t_0}^s \Big(\frac{\partial k(s,\sigma)}{\partial s} + G(s,s,\sigma)\Big)d\sigma + \int_{t_0}^s \int_{t_0}^\sigma \frac{\partial G(s,\sigma,r)}{\partial s}drd\sigma\Big]ds\}.
$$

Now, we derive the error estimate in the weighted L^2_w norm for the proposed numerical Tau method in the previous section.

Theorem 3.4. *Let $\phi(x,t)$ be the exact solution to the fourth-order boundary value PDE problem (1), which is assumed to be sufficiently smooth. Let $\phi_N(x,t)$ be the spectral Legendre Tau approximation of $\phi(x,t)$ which is defined by (22). If $\phi(x,t) \in H^m_w(0,1) \times \mathbb{R}^+$, then for $m \geq 1$,*

$$\begin{aligned} \|\phi - \phi_N\|_{L^2_w(0,1)} \leq & CN^{-m}|f|_{H^{m,N}_w(0,1)} \\ & + CN^{-m}|K_1|_{H^{m,N}_w(0,1)}\Big(\|\frac{\partial^4\phi}{\partial x^4}\|_{L^2_w(0,1)} + CN^{-m}|\frac{\partial^4\phi}{\partial x^4}|_{H^{m,N}_w(0,1)}\Big) \\ & + CN^{-m}|K_2|_{H^{m,N}_w(0,1)}\Big(\|\frac{\partial^5\phi}{\partial t\partial x^4}\|_{L^2_w(0,1)} + CN^{-m}|\frac{\partial^5\phi}{\partial t\partial x^4}|_{H^{m,N}_w(0,1)}\Big) \\ & + CN^{-m}|K_3|_{H^{m,N}_w(0,1)}\Big(\|\frac{\partial^4\phi}{\partial x^4}\|_{L^2_w(0,1)} + CN^{-m}|\frac{\partial^4\phi}{\partial x^4}|_{H^{m,N}_w(0,1)}\Big) \\ & + CN^{-m}|K_4|_{H^{m,N}_w(0,1)}\Big(\|\frac{\partial^5\phi}{\partial t\partial x^4}\|_{L^2_w(0,1)} + CN^{-m}|\frac{\partial^5\phi}{\partial t\partial x^4}|_{H^{m,N}_w(0,1)}\Big), \end{aligned} \tag{25}$$

provided that N is sufficiently large and C is a constant independent of N.

Proof. First, we can consider the equation (2) in the following form

$$\frac{\partial^4\phi}{\partial x^4} = \frac{1}{\beta_2}\frac{\partial\phi}{\partial t} - \frac{\beta_1}{\beta_2}\frac{\partial^2\phi}{\partial x^2}, \qquad x \in I = [0,1],\ t > 0, \tag{26}$$

assume that $\frac{\partial^4\phi_N}{\partial x^4} = u_N(x,t)$. Considering the Dirichlet and Neumann boundary conditions and using (22) and (23), we have

$$\begin{aligned} \phi_N(x,t) = & g_1(t) + x(g_2(t) - g_1(t)) + \frac{1}{2}(x^2 - x)h_1(t) + \frac{1}{6}(x^3 - x)(h_2(t) - h_1(t)) \\ & + \int_0^1 -\frac{1}{6}x\Big((x^2-1)(1-t_1) + (1-t_1)^3\Big)u_N(t_1,t)dt_1 \\ & + \int_0^x \frac{1}{6}(x-t_1)^3 u_N(t_1,t)dt_1, \end{aligned} \tag{27}$$

and

$$\begin{aligned} u_N(x,t) = & f_N(x,t) + \int_0^1 K_{1,N}(x,t_1)u_N(t_1,t)dt_1 + \int_0^1 K_{2,N}(x,t_1)\frac{\partial u_N(t_1,t)}{\partial t}dt_1 \\ & + \int_0^x K_{3,N}(x,t_1)u_N(t_1,t)dt_1 + \int_0^x K_{4,N}(x,t_1)\frac{\partial u_N(t_1,t)}{\partial t}dt_1. \end{aligned} \tag{28}$$

Subtracting (23) from (28) yields

$$\begin{aligned} e(x,t) = & \int_0^1 K_1(x,t_1)e(t_1,t)dt_1 + \int_0^1 K_2(x,t_1)\frac{\partial e(t_1,t)}{\partial t}dt_1 \\ & + \int_0^x K_3(x,t_1)e(t_1,t)dt_1 + \int_0^x K_4(x,t_1)\frac{\partial e(t_1,t)}{\partial t}dt_1 \\ & + I_1 + I_2 + I_3 + I_4 + I_5, \end{aligned} \tag{29}$$

where $e(x,t) = u(x,t) - u_N(x,t)$ is the error function, and

$$I_1 = f(x,t) - f_N(x,t),$$
$$I_2 = \int_0^1 (K_1(x,t_1) - K_{1,N}(x,t_1))u_N(t_1,t)dt_1,$$
$$I_3 = \int_0^1 (K_2(x,t_1) - K_{2,N}(x,t_1))\frac{\partial u_N(t_1,t)}{\partial t}dt_1,$$
$$I_4 = \int_0^x (K_3(x,t_1) - K_{3,N}(x,t_1))u_N(t_1,t)dt_1,$$
$$I_5 = \int_0^x (K_4(x,t_1) - K_{4,N}(x,t_1))\frac{\partial u_N(t_1,t)}{\partial t}dt_1.$$

Then (29) can be written as

$$\begin{aligned} e(x,t) &= \int_0^1 K_1(x,t_1)e(t_1,t)dt_1 + \int_0^1\int_0^t K_2(x,t_1)\big(e(t_1,\tau)d\tau dt_1 \\ &\quad + \int_0^x K_3(x,t_1)e(t_1,t)dt_1 + \int_0^x\int_0^t K_4(x,t_1)e(t_1,\tau)d\tau dt_1 \\ &\quad + I_1 + I_2 + I_3 + I_4 + I_5. \end{aligned} \tag{30}$$

Applying Gronwall's inequality from Theorem 3.3 and using Lemma 3.2 with $p = q = 2$ and $w_1(x) = w_2(x) = 1$ for (30) yields

$$\|e\|_{L^2_w(0,1)} \leq C\|I_1 + I_2 + I_3 + I_4 + I_5\|_{L^2_w(0,1)}. \tag{31}$$

Using the inequality (24) from Lemma 3.1, gives

$$\|I_1\|_{L^2_w(0,1)} \leq CN^{-m}|f|_{H^{m,N}_w(0,1)}. \tag{32}$$

It follows from the Hardy's inequality (see Lemma 3.2) and the inequality (24) that

$$\begin{aligned} \|I_2\|_{L^2_w(0,1)} &\leq \|K_1 - K_{1,N}\|_{L^2_w(0,1)}\|u_N\|_{L^2_w(0,1)} \\ &\leq CN^{-m}|K_1|_{H^{m,N}_w(0,1)}\big(\|u\|_{L^2_w(0,1)} + \|e\|_{L^2_w(0,1)}\big), \\ \|I_3\|_{L^2_w(0,1)} &\leq \|K_2 - K_{2,N}\|_{L^2_w(0,1)}\|\frac{\partial u_N}{\partial t}\|_{L^2_w(0,1)} \\ &\leq CN^{-m}|K_2|_{H^{m,N}_w(0,1)}\big(\|\frac{\partial u}{\partial t}\|_{L^2_w(0,1)} + \|\frac{\partial e}{\partial t}\|_{L^2_w(0,1)}\big), \end{aligned} \tag{33}$$

and similarly

$$\begin{aligned} \|I_4\|_{L^2_w(0,1)} &\leq \|K_3 - K_{3,N}\|_{L^2_w(0,1)}\|u_N\|_{L^2_w(0,1)} \\ &\leq CN^{-m}|K_3|_{H^{m,N}_w(0,1)}\big(\|u\|_{L^2_w(0,1)} + \|e\|_{L^2_w(0,1)}\big), \\ \|I_5\|_{L^2_w(0,1)} &\leq \|K_4 - K_{4,N}\|_{L^2_w(0,1)}\|\frac{\partial u_N}{\partial t}\|_{L^2_w(0,1)} \\ &\leq CN^{-m}|K_4|_{H^{m,N}_w(0,1)}\big(\|\frac{\partial u}{\partial t}\|_{L^2_w(0,1)} + \|\frac{\partial e}{\partial t}\|_{L^2_w(0,1)}\big). \end{aligned} \tag{34}$$

Finally, the above estimates together with (32), (33), (34) and Lemma 3.1, reveal the convergence results of the presented numerical scheme. □

Remark 3.5. Considering Theorem 3.4, the rate of convergence can be obtained N^{-m}. Note that the smoother the solution, the larger the value of m, and therefore, the better the approximation (for some numerical methods, the rate of convergence is fixed regardless of the smoothness of the function). This rate of convergence is referred to in the literature as spectral convergence.

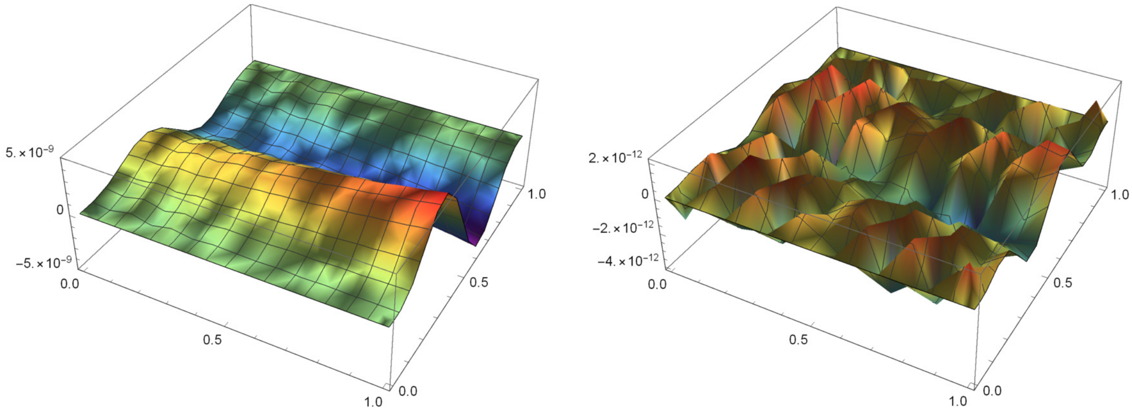

Fig. 1. The plots of error functions for $N = 4$ (left) and $N = 6$ (right) in Example 1 (case 1).

Table 1
$||\phi(x, t_n) - \phi_N(x, t_n)||_{L^2(0,1)}$ for Example 1 in **case 1**.

t_n	$N = 4$	$N = 5$	$N = 6$
0.1	1.20×10^{-9}	2.09×10^{-11}	3.78×10^{-13}
0.2	1.33×10^{-9}	2.31×10^{-11}	4.18×10^{-13}
0.3	1.46×10^{-9}	2.55×10^{-11}	4.62×10^{-13}
0.4	1.62×10^{-9}	2.82×10^{-11}	5.10×10^{-13}
0.5	1.79×10^{-9}	3.12×10^{-11}	5.64×10^{-13}
0.6	1.98×10^{-9}	3.45×10^{-11}	6.23×10^{-13}
0.7	2.19×10^{-9}	3.81×10^{-11}	6.89×10^{-13}
0.8	2.42×10^{-9}	4.21×10^{-11}	7.68×10^{-13}
0.9	2.67×10^{-9}	4.69×10^{-11}	8.41×10^{-13}
Order	14.75	18.11	22.05

4. Numerical experimentation and discussion

In this section, the numerical procedure outlined in Section 2 is applied to solve two specific problems with known exact solutions. We also solve these problems directly by the spectral Tau method to compare and show good numerical stability properties of converting PDEs to integral equations. All computations are performed by the Mathematica software. We use the shifted Legendre polynomials in the interval $[0, 1]$ as orthogonal basis functions.

Example 4.1. Consider problem (1) with

$$
\begin{aligned}
&a = 0, \quad b = 1, \quad \beta_1 = 2, \quad \beta_2 = -1, \\
&\phi(x, 0) = e^x, \quad \phi(0, t) = e^t, \qquad \phi(1, t) = e^{1+t}, \\
&\frac{\partial^2 \phi(0, t)}{\partial x^2} = e^t, \quad \frac{\partial^2 \phi(1, t)}{\partial x^2} = e^{1+t},
\end{aligned}
$$

where the exact solution $\phi(x, t) = e^{x+t}$.

Now, we consider two structures to find approximate solution to this problem:

Case 1: According to the proposed method in section 2, we first convert this problem to Volterra-Fredholm integral equation (12) and then seek an approximate solution by the Legendre spectral Tau method. Also, for the solving obtained initial-value problem, we choose the Gauss points ($c_1 = \frac{1}{10}(5 - \sqrt{15}), c_2 = \frac{1}{2}, c_3 = \frac{1}{10}(5 + \sqrt{15})$) as collocation parameters with the stepsize $h = 0.01$ for interval $[0, 1]$. Errors in $L^2(0, 1)$-norm have been shown for different values of N at some gridpoints of t_n in Table 1. Also, using maximum errors at some gridpoints of t_n, we report the convergence order of the Tau-Legendre method which displays the exponential rate of convergence for the problem with complete smooth solution.

Case 2: In this case, we apply the Legendre spectral Tau method to solve this PDE problem directly (without converting to the integral equation) and show the errors in $L^2(0, 1)$-norm for different values of N at some gridpoints t_n in Table 2.

Table 2
$||\phi(x,t_n) - \phi_N(x,t_n)||_{L^2(0,1)}$ for Example 4.1 in **case 2**.

t_n	$N=4$	$N=5$	$N=6$
0.1	3.77×10^{-4}	2.65×10^{-4}	2.25×10^{-6}
0.2	4.16×10^{-4}	2.93×10^{-4}	2.48×10^{-6}
0.3	4.60×10^{-4}	3.24×10^{-4}	2.74×10^{-6}
0.4	5.08×10^{-4}	3.58×10^{-4}	3.03×10^{-6}
0.5	5.62×10^{-4}	3.96×10^{-4}	3.35×10^{-6}
0.6	6.21×10^{-4}	4.27×10^{-4}	3.71×10^{-6}
0.7	6.87×10^{-4}	4.84×10^{-4}	4.10×10^{-6}
0.8	7.59×10^{-4}	5.34×10^{-4}	4.53×10^{-6}
0.9	8.39×10^{-4}	5.91×10^{-4}	5.00×10^{-6}

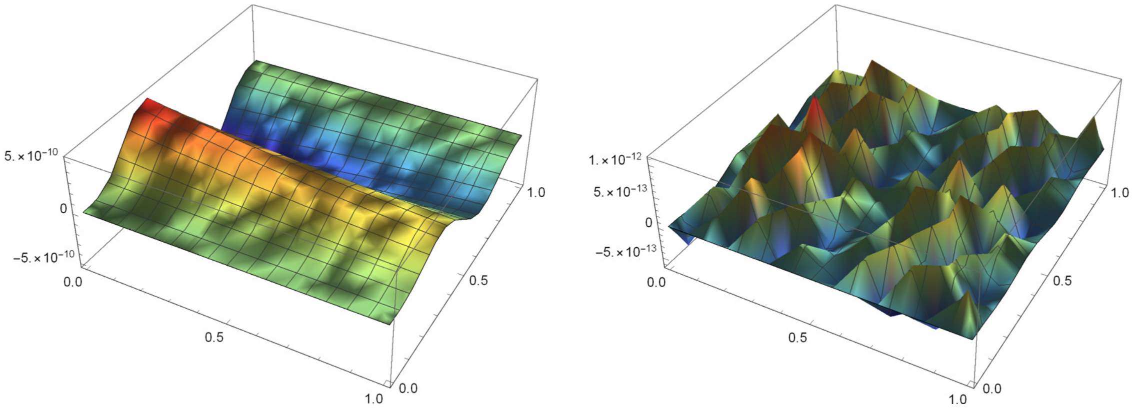

Fig. 2. The plots of error functions for $N=4$ (left) and $N=6$ (right) in Example 2 (case 1).

Table 3
$||\phi(x,t_n) - \phi_N(x,t_n)||_{L^2(0,1)}$ for Example 4.2 in **case 1**.

t_n	$N=4$	$N=5$	$N=6$
0.1	2.90×10^{-10}	9.14×10^{-12}	9.00×10^{-14}
0.2	2.63×10^{-10}	8.27×10^{-12}	8.13×10^{-14}
0.3	2.38×10^{-10}	7.49×10^{-12}	7.36×10^{-14}
0.4	2.15×10^{-10}	6.77×10^{-12}	6.66×10^{-14}
0.5	1.94×10^{-10}	6.13×10^{-12}	6.02×10^{-14}
0.6	1.76×10^{-10}	5.54×10^{-12}	5.45×10^{-14}
0.7	1.59×10^{-10}	5.02×10^{-12}	4.93×10^{-14}
0.8	1.44×10^{-10}	4.54×10^{-12}	4.46×10^{-14}
0.9	1.30×10^{-10}	4.11×10^{-12}	4.03×10^{-14}
Order	17.29	19.04	20.99

Example 4.2. Consider problem (1) with

$$
\begin{aligned}
&a=0, \quad b=1, \quad \beta_1=0, \quad \beta_2=-1,\\
&\phi(x,0)=-\cos x, \quad \phi(0,t)=-e^{-t}, \qquad \phi(1,t)=-e^{-t}\cos 1,\\
&\frac{\partial^2\phi(0,t)}{\partial x^2}=e^{-t}, \quad \frac{\partial^2\phi(1,t)}{\partial x^2}=e^{-t}\cos 1,
\end{aligned}
$$

where the exact solution $\phi(x,t)=-e^{-t}\cos x$.

Similar to the Example 4.1, the proposed methods in two cases have been implemented for this problem and the $L^2(0,1)$ errors for different values of N at the gridpoints t_n have been reported in Table 3, 4.

In the Examples 4.1 and 4.2 the solutions are smooth. It is better that we have an example with lower regularity. For this purpose, let pay attention to the following remark.

Remark 4.3. By considering the equation (1) with $\beta_2=0$, which the error function $erf(\frac{1-x}{2(\sqrt{1+t})})$ is the exact solution. In this example, according to the process of section 2, we assume $\frac{\partial^2\phi(x,t)}{\partial x^2}=u(x,t)$ then corresponding to the relation (12), we obtain:

Table 4
$||\phi(x,t_n)-\phi_N(x,t_n)||_{L^2(0,1)}$ for Example 4.2 in **case 2**.

t_n	$N=4$	$N=5$	$N=6$
0.1	1.44×10^{-4}	1.30×10^{-4}	9.37×10^{-7}
0.2	1.30×10^{-4}	1.18×10^{-4}	4.87×10^{-7}
0.3	1.18×10^{-4}	1.07×10^{-4}	4.41×10^{-7}
0.4	1.07×10^{-4}	9.69×10^{-5}	3.98×10^{-7}
0.5	9.69×10^{-5}	8.76×10^{-5}	3.60×10^{-7}
0.6	8.77×10^{-5}	7.93×10^{-5}	3.26×10^{-7}
0.7	7.93×10^{-5}	7.17×10^{-5}	2.96×10^{-7}
0.8	7.18×10^{-5}	6.49×10^{-5}	2.67×10^{-7}
0.9	6.49×10^{-5}	5.87×10^{-5}	2.45×10^{-7}

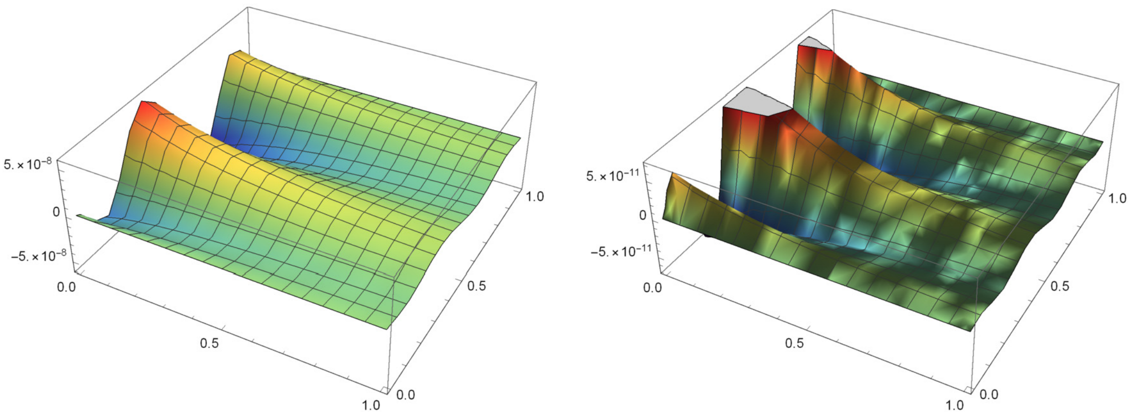

Fig. 3. The plots of error functions for $N=4$ (left) and $N=6$ (right) in Example 3 (case 1).

Table 5
$||\phi(x,t_n)-\phi_N(x,t_n)||_{L^2(0,1)}$ for Example 4.4 in **case 1**.

t_n	$N=4$	$N=5$	$N=6$
0.1	2.62×10^{-8}	1.59×10^{-9}	5.44×10^{-11}
0.2	2.02×10^{-8}	1.09×10^{-9}	3.92×10^{-11}
0.3	1.58×10^{-8}	7.25×10^{-10}	2.88×10^{-11}
0.4	1.26×10^{-8}	5.62×10^{-10}	2.15×10^{-11}
0.5	1.01×10^{-8}	4.16×10^{-10}	1.63×10^{-11}
0.6	8.29×10^{-9}	3.14×10^{-10}	1.26×10^{-11}
0.7	6.83×10^{-9}	2.41×10^{-10}	9.87×10^{-12}
0.8	5.69×10^{-9}	1.87×10^{-10}	7.81×10^{-12}
0.9	4.78×10^{-9}	1.48×10^{-10}	6.25×10^{-12}

$$\phi(x,t)=g_1(t)+\frac{x}{b}(g_2(t)-g_1(t))-\frac{x}{b}\int_0^b(b-t_1)u(t_1,t)dt_1+\int_0^x(x-t_1)u(t_1,t)dt_1, \tag{35}$$

and

$$\begin{aligned}u(x,t)=\hat{f}(x,t)&+\int_a^b K_1(x,t_1)u(t_1,t)dt_1+\int_a^b K_2(x,t_1)\frac{\partial u(t_1,t)}{\partial t}dt_1\\&+\int_a^x K_3(x,t_1)u(t_1,t)dt_1+\int_a^x K_4(x,t_1)\frac{\partial u(t_1,t)}{\partial t}dt_1,\end{aligned} \tag{36}$$

where

$$\begin{aligned}&\hat{f}(x,t)=\frac{1}{\beta_1}(g_1'(t)+\frac{x}{b}(g_2'(t)-g_1'(t)),\\&K_1(x,t_1)=0, \qquad K_2(x,t_1)=-\tfrac{1}{\beta_1}(b-t)x,\\&K_3(x,t_1)=0, \qquad K_4(x,t_1)=\tfrac{1}{\beta_1}(x-t_1).\end{aligned}$$

Table 6
$||\phi(x,t_n)-\phi_N(x,t_n)||_{L^2(0,1)}$ for Example 4.4 in **case 2**.

t_n	$N=4$	$N=5$	$N=6$
0.1	1.22×10^{-5}	5.45×10^{-7}	2.63×10^{-8}
0.2	1.01×10^{-5}	4.17×10^{-7}	2.05×10^{-8}
0.3	8.50×10^{-6}	3.26×10^{-7}	1.60×10^{-8}
0.4	7.20×10^{-6}	2.54×10^{-7}	1.28×10^{-8}
0.5	6.16×10^{-6}	2.00×10^{-7}	1.22×10^{-8}
0.6	5.32×10^{-6}	1.61×10^{-7}	2.03×10^{-8}
0.7	4.63×10^{-6}	1.30×10^{-7}	9.85×10^{-9}
0.8	4.06×10^{-6}	1.07×10^{-7}	6.02×10^{-9}
0.9	3.58×10^{-6}	9.01×10^{-8}	4.99×10^{-9}

Example 4.4. In this example we consider relations (35) and (36) respectively, instead of (10) and (12) for the problem (1) with

$$a=0,\qquad b=1,\quad \beta_1=1,\qquad \beta_2=0,$$
$$\phi(x,0)=erf(\frac{1-x}{2}),\quad \phi(0,t)=erf(\frac{1}{2(\sqrt{1+t})}),\qquad \phi(1,t)=0,$$
$$\frac{\partial^2\phi(0,t)}{\partial x^2}=-\frac{e^{-\frac{1}{4(1+t)}}}{2\sqrt{\pi}(1+t)^{\frac{3}{2}}},\quad \frac{\partial^2\phi(1,t)}{\partial x^2}=0,$$

where the exact solution $\phi(x,t)=erf(\frac{1-x}{2(\sqrt{1+t})})$.

Similar to the Examples 4.1 and 4.2, the proposed methods in two cases have been implemented for this problem and the $L^2(0,1)$ errors for different values of N at the gridpoints t_n have been reported in Table 5, 6. Plots of the error functions for $N=4,6$ in the Examples 4.1, 4.2, 4.4 (**case 1**) are also given in Figs. 1, 2 and 3. From the Tables 1, 3, 5 and 2, 4, 6 we observe that the results obtained by **case 1** are more accurate than the direct spectral Tau method in **case 2** to solve these Examples.

5. Conclusion

This paper proposes a numerical method for the fourth-order partial differential equation with boundary conditions based on spectral methods. We transformed this partial differential equation into the Volterra-Fredholm integral equation to investigate the convergence analysis of the proposed numerical method. The most important contribution of this work is that the result obtained from using the Tau method to solve the converted problem (Volterra-Fredholm integral equation) is more significant compared to the solving of PDE problem directly by the Tau method.

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