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A Fixed Point Approach to Approximate Bi-Homomorphisms and Bi-Derivations in Intuitionistic Fuzzy Normed Spaces

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Abstract In this paper, we define bi-homomorphisms and bi-derivation in intuitionistic fuzzy normed spaces (IFNspaces). The fixed point methods are implemented to generalized Hyers-Ulam stability of bi-homomorphisms and biderivations associated to the following 2-dimensional vector variable quadratic functional equation in intuitionistic fuzzy normed spaces

$$f(x+y, z+w) + f(x-y, z-w) = 2f(x,z) + 2f(y,w),$$

where $f(x,0) = f(0,y) = 0$ for all $x, y \in X$.

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1. Introduction

The fuzzy set theory is a powerful hand set for modeling uncertainty and vagueness in various problems arising in the field of science and engineering. It has also very useful applications in various field, e.g. population dynamic [1], chaos control [2, 3], computer programming [4], nonlinear dynamical systems [5], nonlinear operators [6], statistical coverage [7], etc. The fuzzy topology proves to be a very useful tool to deal with such situations where the use of classical theories breaks down.

In 1984, Katsaras [8] defined fuzzy norms on linear spaces and the same year Wu and Fang also introduced a notion of fuzzy normed spaces and gave the generalization of the Kolmogoroff normalized theorem for fuzzy topological linear spaces. In [9], Biswas defined and studied fuzzy inner products on linear spaces. Since then some mathematicians have defined fuzzy metrics and norms on linear spaces from various points of view [10–14].

In 1994, Cheng and Mordeson introduced a definition of fuzzy norm on linear spaces in such a manner that the corresponding induced fuzzy metrics are of Kramosil and Michalek

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type. In 2003, Bag and Samanta [15] modified the definition of Cheng and Mordeson by removing a regular condition.

Let X be a real linear space. A function $N : X \times \mathbb{R} \to [0, 1]$ (the so-called fuzzy subset) is said to be a fuzzy norm on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,

$(N1)$ $N(x, c) = 0$ for $c \leqslant 0$;

$(N2)$ $x = 0$ if and only if $N(x, c) = 1$ for all $c > 0$;

$(N3)$ $N(cx, t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;

$(N4)$ $N(x + y, s + t) \geqslant \min\{N(x, s), N(y, t)\}$;

$(N5)$ $N(x, .)$ is a non-decreasing function on $\mathbb{R}$ and $\lim_{t\to\infty} N(x, t) = 1$;

$(N6)$ For $x \neq 0, N(x, .)$ is continuous on $\mathbb{R}$.

The pair (X, N) is called a *fuzzy normed linear space*. One may regard $N(x, t)$ as the truth value of the statement "the norm of x is less than or equal to the real number t".

There are many situations where the norm of a vector is not possible to find and the concept of intuitionistic fuzzy norm [16–18] seems to be more suitable in such cases, that is, we can deal with such situations by modelling the inexactness through the intuitionistic fuzzy norm. Stability problem of a functional equation was first posed by Ulam [19] which was answered by Hyers [20] and then generalized by Aoki [21] and Rassias [22] for additive mappings and linear mappings, respectively. Since then several stability problems for various functional equations have been investigated in [23, 24]; various fuzzy stability results concerning Cauchy, Jensen, quadratic and cubic functional equations were discussed in [25–29]; and some random stability results concerning Jensen and cubic functional equations were discussed in [30, 31]. The stability problem for the 2-dimensional vector variable quadratic functional equation was proved by the authors [32] for mappings $f : X \times X \to Y$, where X is a real normed space and Y is a Banch space.

In this paper, we determine the stability of bi-homomorphisms and bi-derivations concerning the following 2-dimentional vector variable quadratic functional equation in intuitionistic fuzzy normed spaces

$$f(x + y, z + w) + f(x - y, z - w) = 2f(x, z) + 2f(y, w), \tag{1.1}$$

where $f(x, 0) = f(0, y) = 0$ for all $x, y \in X$.

2. Preliminaries

In this section we recall some terminology, notation and definitions used in this paper [16–18].

Consider the set L^* and the order relation $\leqslant_{L^*}$ defined by

$$L^* = \{(x_1, x_2) : (x_1, x_2) \in [0, 1]^2, x_1 + x_2 \leqslant 1\},$$
$$(x_1, x_2) \leqslant_{L^*} (y_1, y_2) \Leftrightarrow x_1 \leqslant y_1,\ x_2 \geqslant y_2,\ (x_1.x_2), (y_1, y_2) \in L^*.$$

Then $(L^*, \leqslant_{L^*})$ is a complete lattice ([18]). We denote its unite by $0_{L^*} = (0, 1)$ and $1_{L^*} = (1, 0)$.

Definition 2.1. Let U be a non-empty set called the universe. An L^*-fuzzy set in U is defined as a mapping $\mathcal{A} : U \to L^*$. For each u in U, $\mathcal{A}(u)$ represents the degree (in L^*) to which u is an element of U. An intuitionstic fuzzy set $\mathcal{A}_{\zeta,\eta}$ in a universal set U is an object $\mathcal{A}_{\zeta,\eta} = \{(\zeta_{\mathcal{A}}(u), \eta_{\mathcal{A}}(u)) : u \in U\}$, where $\zeta_{\mathcal{A}}(u) \in [0, 1]$ and $\eta_{\mathcal{A}}(u) \in [0, 1]$ for all $u \in U$ are called the membership degree and the non-membership degree, respectively, of u in $\mathcal{A}_{\zeta,\eta}$ and, furthermore, satisfy $\zeta_{\mathcal{A}}(u) + \eta_{\mathcal{A}}(u) \leqslant 1$.

Definition 2.2. A triangular norm (t-norm) on L^* is mapping $\mathcal{T} : (L^*)^2 :\to L^*$ satisfying the following conditions:
(a) $(\forall x \in L^*)(\mathcal{T}(x, 1_{L^*}) = x)$ (boundary condition);
(b) $(\forall (x, y) \in (L^*)^2)(\mathcal{T}(x, y) = \mathcal{T}(y, x))$ (commutativity);
(c) $(\forall (x, y, z) \in (L^*)^3)(\mathcal{T}(x, \mathcal{T}(y, z)) = \mathcal{T}(\mathcal{T}(x, y), z))$ (associativity);
(d) $(\forall (x, x', y, y') \in (L^*)^4)(x \leqslant_{L^*} x'$ and $y \leqslant_{L^*} y' \Rightarrow \mathcal{T}(x, y) \leqslant_{L^*} \mathcal{T}(x', y'))$ (monotonicity).

A t-norm $\mathcal{T}$ on L^* is said to be continuous if, for any $x, y \in L^*$ and any sequences $\{x_n\}$ and $\{y_n\}$ which converge to x and y, respectively,

$$\lim_{n\to\infty} \mathcal{T}(x_n, y_n) = \mathcal{T}(x, y).$$

For examples, let $a = (a_1, a_2), b = (b_1, b_2) \in L^*$, consider $\mathcal{T}(a, b) = (a_1 b_1, \min\{a_2 + b_2, 1\})$ and
$\mathcal{M}(a, b) = (\min\{a_1, b_1\}, \max\{a_2, b_2\})$. Then $\mathcal{T}(a, b)$ and (a, b) are continuous t-norm.

Now, we define a sequence $\mathcal{T}^n$ recursively by $\mathcal{T}^1 = \mathcal{T}$ and

$$\mathcal{T}^n(x^{(1)}, \cdots, x^{(n+1)}) = \mathcal{T}(\mathcal{T}^{n-1}(x^{(1)}, \cdots, x^{(n)}), x^{(n+1)})$$

for all $n \geqslant 2$ and $x^{(i)} \in L^*$.

Definition 2.3. A negator on L^* is any decreasing mapping $\mathcal{N} : L^* :\to L^*$ satisfying $\mathcal{N}(0_{L^*}) = 1_{L^*}$ and $\mathcal{N}(1_{L^*}) = 0_{L^*}$. If $\mathcal{N}(\mathcal{N}(x)) = x$ for all $x \in L^*$, then $\mathcal{N}$ is called an involutive negator. A negator on $[0, 1]$ is a decreasing mapping $N : [0, 1] \to [0, 1]$ satisfying $N(0) = 1$ and $N(1) = 0$. N_s denotes the standard negator on $[0, 1]$ define by $N_s(x) = 1 - x$ for all $x \in [0, 1]$.

The definition of an intuitionistic fuzzy normed space is given below (see [18]).

Definition 2.4. (1) Let $\mathcal{L} = (L^*, \leqslant_{L^*})$. The triple $(X, \mathcal{P}, \mathcal{T})$ is said to be an $\mathcal{L}$-fuzzy normed space if X is a vector space, $\mathcal{T}$ is a continuous t-norm on L^* and P is an L^*-fuzzy set on $X \times (0, +\infty)$ satisfying the following conditions for all $x, y \in X$ and $t, s > 0$,
(a) $\mathcal{P}(x, t) >_{L^*} 0_{L^*}$;
(b) $\mathcal{P}(x, t) = 1_{L^*}$ if and only if $x = 0$;
(c) $\mathcal{P}(\alpha x, t) = \mathcal{P}(x, \frac{x}{|\alpha|})$ for all $\alpha \neq 0$;
(d) $\mathcal{P}(x + y, t + s) \geqslant_{L^*} \mathcal{T}(\mathcal{P}(x, t), \mathcal{P}(y, s))$;
(e) $\mathcal{P}(x, .) : (0, \infty) \to L^*$ is continuous;
(f) $\lim_{t\to 0} \mathcal{P}(x, t) = 0_{L^*}$ and $\lim_{t\to\infty} \mathcal{P}(x, t) = 1_{L^*}$.

In this case $\mathcal{P}$ is called an $\mathcal{L}$-fuzzy norm (briefly, L^*-fuzzy norm).

(2) If $\mathcal{P} = \mathcal{P}_{\mu,\nu}$ is an intuitionistic fuzzy set (see Definition 2.1), then the triple $(X, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ is said to be an intuitionstic fuzzy normed space (briefly, IFN-space). In this case $\mathcal{P} = \mathcal{P}_{\mu,\nu}$ is said an intuitionistic fuzzy norm on X.

Note that, if $\mathcal{P}$ is an L^*-fuzzy norm on X, then the following are satisfied:
(i) $\mathcal{P}(x, t)$ is nondecreasing with respect to t for all $x \in X$.
(ii) $\mathcal{P}(x - y, t) = \mathcal{P}(y - x, t)$ for all $x, y \in X$ and $t > 0$(see [18]).

Example 2.5. Let $(X, \|.\|)$ be a normed space. Let $\mathcal{T}(a, b) = (a_1 b_1, \min(a_2 + b_2, 1))$ for all $a = (a_1, a_2), b = (b_1, b_2) \in L^*$ and μ, ν be membership and non-membership degree of an intuitionistic fuzzy set defined by

$$\mathcal{P}_{\mu,\nu}(x, t) = (\mu_x(t), \nu_x(t)) = \Big(\frac{t}{t + m\|x\|}, \frac{\|x\|}{t + \|x\|}\Big)$$

for all $t \in \mathbb{R}^+$ in which $m > 1$. Then $(X, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ is an IFN-space. Here, $\mu(x,t)+\nu(x,t) = 1$ for $x = 0$ and $\mu(x,t) + \nu(x,t) < 1$ for $x \neq 0$.

Let $\mathcal{M}(a,b) = (\min\{a_1, b_1\}, \max\{a_2, b_2\})$ for all $a = (a_1, a_2), b = (b_1, b_2) \in L^*$ and μ, ν be membership and non-membership degree of an intuitionistic fuzzy set defined by

$$\mathcal{P}_{\mu,\nu}(x,t) = (\mu_x(t), \nu_x(t)) = \Big(e^{-\|x\|/t}, e^{-\|x\|/t}(e^{\|x\|/t} - 1)\Big)$$

for all $x \in X$ and $t \in \mathbb{R}^+$. Then $(X, \mathcal{P}_{\mu,\nu}, \mathcal{M})$ is an IFN-space.

Definition 2.6. (1) A sequence $\{x_n\}$ in an IFN-space $(X, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ is said to be convergent to a point $x \in X$ (denoted by $x_n \to x$) if $\mathcal{P}_{\mu,\nu}(x_n - x, t) \to 1_{L^*}$ as $n \to \infty$ for every $t > 0$.

(2) A sequence $\{x_n\}$ in an IFN-space $(X, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ is said to be Cauchy sequence if, for any $0 < \varepsilon < 1$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\mathcal{P}_{\mu,\nu}(x_n - x_m, t) >_{L^*} (N_s(\varepsilon), t), \quad \forall n, m \geqslant n_0,$$

where N_s is the standard negator.

(3) An IFN-space $(X, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ is said to be completed if every Cauchy sequence in $(X, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ is convergent in $(X, \mathcal{P}_{\mu,\nu}, \mathcal{T})$. A complete intuitionistic fuzzy normed space is called an intuitionistic fuzzy Banach space.

The concepts of convergence and Cauchy sequences in an intuitionistic fuzzy normed space are studied in [17].

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a generalized metric on X if d satisfies

(1) $d(x,y) = 0$ if and only if $x = y$;
(2) $d(x,y) = d(y,x)$ for all $x, y \in X$;
(3) $d(x,y) \leqslant d(x,z) + d(y,z)$ for all $x, y, z \in X$.

For explicitly later use, we recall a fundamental result in fixed point theory.

Theorem 2.7. (The fixed point alternative theorem [33]) *Let (Ω, d) be a complete generalized metric space and $J : \Omega \to \Omega$ be a strictly contractive mapping with Lipschitz constant $0 < L < 1$, that is,*

$$d(Jx, Jy) \leqslant Ld(x,y), \quad \forall x, y \in \Omega.$$

Then, for each given $x \in \Omega$, either

$$d(J^n x, J^{n+1}x) = \infty, \quad \forall n \geqslant 0,$$

or

$$d(J^n x, J^{n+1}x) < \infty, \quad \forall n \geqslant n_0,$$

for some nonnegative integer n_0. Actually, if second alternative holds, then the sequence $\{J^n x\}$ converges to a fixed point y^* of J and

(1) y^* is the unique fixed point of J in the set $\Delta = \{y \in \Omega : d(J^{n_0}x, y) < \infty\}$;
(2) $d(y, y^*) \leqslant \frac{1}{1-L} d(y, Jy)$ for all $y \in \Delta$.

Definition 2.8. Let X be a ternary algebra and $(X, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ be an IFN-space.

(1) The intuitionistic fuzzy normed space $(X, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ is called a ternary IFN-space. if

$$\mathcal{P}_{\mu,\nu}([xyz], stu) \geqslant \mathcal{T}^3(\mathcal{P}_{\mu,\nu}(x,s), \mathcal{P}_{\mu,\nu}(y,t), \mathcal{P}_{\mu,\nu}(z,u))$$

for all $x, y, z \in X$ and $s, t, u > 0$.

(2) A complete ternary IFN-space is called a ternary intuitionistic fuzzy Banach space.

Example 2.9. Let $(X, \|.\|)$ be a normed space. Let $\mathcal{T}(a,b) = (a_1 b_1, \min(a_2 + b_2, 1))$ for all $a = (a_1, a_2), b = (b_1, b_2) \in L^*$ and μ, ν be membership and non-membership degree of an intuitionistic fuzzy set defined by

$$\mathcal{P}_{\mu,\nu}(x,t) = (\mu_x(t), \nu_x(t)) = \Big(\frac{t}{t+\|x\|}, \frac{\|x\|}{t+\|x\|}\Big)$$

for all $t \in \mathbb{R}^+$. Then $(X, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ is a ternary fuzzy normed (Banach) space. Here, $\mu(x,t) + \nu(x,t) = 1$ for $x = 0$ and $\mu(x,t) + \nu(x,t) < 1$ for $x \neq 0$.

Definition 2.10. Let X be a ternary normed (Banach) space and $(Y, \mathcal{P}_{\mu,\nu})$ is ternary intuitionistic fuzzy Banach space.
(1) A $\mathbb{C}$-bilinear mapping $H : X \times X \to Y$ is called a ternary bi-homomorphism if

$$H([xyz], w^3) = [H(x,w)H(y,w)H(z,w)],$$
$$H(x^3, [yzw]) = [H(x,y)H(x,z)H(x,w)]$$

for all $x, y, z, w \in X$.
(2) A $\mathbb{C}$-bilinear mapping $\delta : X \times X \to Y$ is called a ternary bi-derivation if

$$\delta([xyz], w) = [\delta(x,w)yz] + [x\delta(y,w)z] + [xy\delta(z,w)],$$
$$\delta(x, [yzw]) = [\delta(x,y)zw] + [y\delta(x,z)w] + [yz\delta(x,w)]$$

for all $x, y, z, w \in X$.

3. Intuitionistic Fuzzy Bi-Homomorphism Stability of Quadratic Functional Equation

In this section, we prove the generalized Ulam-Hyers stability of bi-homomorphism of quadratic functional equation in intuitionistic fuzzy normed spaces, based on the fixed point method. For notational convenience, given a function $f : X \times X \to Y$, we define the difference operator

$$D_q f(x,y,z,w) = f(x+y, z+w) + f(x-y, z-w) - 2f(x,z) - 2f(y,w) \tag{3.1}$$

We begin with a generalized Hyers-Ulam type theorem in IFN-space for the functional equation (1.1).

Theorem 3.1. *Let X be a linear space and let $(Z, \mathcal{P}'_{\mu,\nu}, \mathcal{T}')$ be an IFN-space. Let $\varphi : X^4 \to Z$ be a function, for some real number with $\alpha < 4$, such that*

$$\mathcal{P}'_{\mu,\nu}(\varphi(2x,2y,2z,2w), t) \geqslant_{L^*} \mathcal{P}'_{\mu,\nu}(\alpha\varphi(x,y,z,w), t) \tag{3.2}$$

for all $x, y, z, w \in X$ and all $t > 0$. Let $(Y, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ be an intuitionistic fuzzy Banach space and let $f : X \times X \to Y$ be a mapping, with $f(0,0) = 0$, such that

$$\begin{cases} \mathcal{P}_{\mu,\nu}(f([xyz], w^3) - [f(x,y)f(y,w)f(z,w)], t) \\ \quad +\mathcal{P}_{\mu,\nu}(f(x^3, [yzw]) - [f(x,y)f(x,z)f(x,w)], t) \geqslant_{L^*} \mathcal{P}'_{\mu,\nu}(\varphi(x,y,z,w), t) \end{cases} \tag{3.3}$$

and

$$\mathcal{P}'_{\mu,\nu}(D_q f(x,y,z,w), t) \geqslant_{L^*} \mathcal{P}'_{\mu,\nu}(\varphi(x,y,z,w), t) \tag{3.4}$$

for all $x, y, z, w \in X$ and all $t > 0$. Then there exists a unique intuitionistic fuzzy bi-homomorphism $H : X \times X \to Y$ satisfying (1.1) such that

$$\mathcal{P}_{\mu,\nu}\Big(H(x,y) - f(x,y), t\Big) \geqslant_{L^*} \mathcal{P}'_{\mu,\nu}(\varphi(x,x,y,y), (4-\alpha)t), \tag{3.5}$$

for all $x, y \in X$ and all $t > 0$.

Proof. Put $y = x$ and $w = z$ in (3.2) to obtain

$$\mathcal{P}_{\mu,\nu}(f(2x,2z) - 4f(x,z), t) \geqslant_{L^*} \mathcal{P}'_{\mu,\nu}(\varphi(x,x,z,z), t),$$

for all $x, z \in X$ and all $t > 0$. Replacing y by z in above inequality, we get

$$\mathcal{P}_{\mu,\nu}(f(2x,2y) - 4f(x,y), t) \geqslant_{L^*} \mathcal{P}'_{\mu,\nu}(\varphi(x,x,y,y), t), \tag{3.6}$$

for all $x, y \in X$ and all $t > 0$.

Consider the set $\Omega := \{g : X \times X \to Y\}$ and introduce a complete generalized metric on Ω (see [33]) (as usual, $\inf \emptyset = \infty$):

$$d(g,h) = \inf\{K \in \mathbb{R}^+ : \mathcal{P}_{\mu,\nu}(g(x,y) - h(x,y), Kt) \geqslant_{L^*} \mathcal{P}'_{\mu,\nu}(\varphi(x,x,y,y), t), \forall x, y \in X, t > 0\}. \tag{3.7}$$

Now, we consider the mapping $J : \Omega \to \Omega$ such that

$$J(g(x,y)) = \frac{1}{4} g(2x,2y) \tag{3.8}$$

for all $x, y \in X$ and we prove that J is a strictly contractive mapping of Ω with the Lipschitz constant $\frac{\alpha}{4}$.

Let $g, h \in \Omega$ be given such that $d(g,h) = \varepsilon$. Then

$$\mathcal{P}_{\mu,\nu}(g(x,y) - h(x,y), \varepsilon t) \geqslant_{L^*} \mathcal{P}'_{\mu,\nu}(\varphi(x,x,y,y), t) \tag{3.9}$$

for all $x, y \in X, t > 0$. Hence

$$\begin{aligned}\mathcal{P}_{\mu,\nu}(Jg(x,y) - Jh(x,y)), \varepsilon t) &\geqslant_{L^*} \mathcal{P}_{\mu,\nu}(g(2x,2y) - h(2x,2y)), 4\varepsilon t) \\ &\geqslant_{L^*} \mathcal{P}'_{\mu,\nu}(\varphi(x,x,y,y), \frac{4t}{\alpha})\end{aligned} \tag{3.10}$$

for all $x, y \in X$ and $t > 0$. By definition, $d(Jg, Jh) \leqslant \frac{\alpha}{4}\varepsilon$. Therefore,

$$d(Jg, Jh) \leqslant \frac{\alpha}{4} d(g,h) \quad \textit{for all } g, h \in \Omega. \tag{3.11}$$

It follows from (3.6) that $d(f, Jf) \leqslant \frac{1}{4}$. Therefore, by Theorem 2.7, there exists a mapping $H : X \to Y$ satisfying:

(1) H is a fixed point of J, that is

$$H(2x,2y) = 4H(x,y) \quad \textit{for all } x, y \in X. \tag{3.12}$$

The mapping H is a unique fixed point of J in the set $\Delta = \{g \in \Omega : d(g,f) < \infty\}$. This implies that H is a unique mapping satisfying (3.5) such that there exists a $K > 0$ satisfying

$$\mathcal{P}_{\mu,\nu}(f(x,y) - H(x,y), Kt) \geqslant_{L^*} \mathcal{P}'_{\mu,\nu}(\varphi(x,x,y,y), t) \tag{3.13}$$

for all $x, y \in X$ and $t > 0$;

(2) $d(J^n f, H) \to 0$ as $n \to \infty$. This implies the equality

$$H(x,y) := \lim_{n\to\infty} J^n f(x,y) = \lim_{n\to\infty} \frac{1}{4^n} f(2^n x, 2^n y) \tag{3.14}$$

for all $x, y \in X$;

(3) $d(f, H) \leqslant \frac{1}{1-L} d(f, Jf)$ with $f \in \Delta$, which implies the inequality

$$d(f, H) \leqslant \frac{1}{4-\alpha}, \tag{3.15}$$

from which it follows

$$\mathcal{P}_{\mu,\nu}(H(x,y) - f(x,y), \frac{t}{4-\alpha}) \geqslant_{L^*} (\varphi(x,x,y,y), t) \tag{3.16}$$

for all $x, y \in X$ and $t > 0$. This implies that the inequality (3.5) holds.

It remains to show that H is a quadratic map. Replacing x and y by $2^n x$ and $2^n y$ in (3.4), respectively, it follows that

$$\begin{aligned}\mathcal{P}_{\mu,\nu}(4^{-n} D_q f(2^n x, 2^n y, 2^n z, 2^n w), t) &\geqslant_{L^*} \mathcal{P'}_{\mu,\nu}(\varphi(2^n x, 2^n y, 2^n z, 2^n w), 4^n t) \\ &\geqslant_{L^*} \mathcal{P'}_{\mu,\nu}(\varphi(x,y,z,w), \frac{4^n t}{\alpha^n}).\end{aligned}$$

Taking the limit as $n \to \infty$, we find that H fulfills (1.1).

From (3.3) and definition of H, we can write

$$\begin{aligned}&\mathcal{P}_{\mu,\nu}(H([xyz], w^3) - [H(x,y)f(y,w)H(z,w)], t) + \mathcal{P}_{\mu,\nu}(H(x^3, [yzw]) \\ &\quad - [H(x,y)H(x,z)H(x,w)], t) \\ &= \mathcal{P}_{\mu,\nu}\Big(\frac{f([2^n x 2^n y 2^n z], (2^n w)^3)}{64^n} - \Big[\frac{f(2^n x, 2^n w)}{4^n}\frac{f(2^n y, 2^n w)}{4^n}\frac{f(2^n z, 2^n w)}{4^n}\Big], t\Big) \\ &\quad + \mathcal{P}_{\mu,\nu}\Big(\frac{f((2^n x)^3, [2^n y 2^n z 2^n w])}{64^n} - \Big[\frac{f(2^n x, 2^n y)}{4^n}\frac{f(2^n x, 2^n z)}{4^n}\frac{f(2^n x, 2^n w)}{4^n}\Big], t\Big) \\ &\geqslant_{L^*} \mathcal{P'}_{\mu,\nu}(\varphi(2^n x, 2^n y, 2^n z, 2^n w), 4^{3n} t) \geqslant_{L^*} \mathcal{P'}_{\mu,\nu}\Big(\varphi(x,y,z,w), \frac{4^{3n} t}{\alpha^n}\Big) \to 1\end{aligned}$$

as $n \to \infty$ for all $x, y, z, w \in X$ and all $t > 0$, because of $\lim_{n\to\infty} \frac{4^{3n} t}{\alpha^n} = \infty$. Therefore it is concluded that

$$\begin{aligned}H([xyz], w^3) &= [H(x,w)H(y,w)H(z,w)], \\ H(x^3, [yzw]) &= [H(x,y)H(x,z)H(x,w)].\end{aligned}$$

To prove the uniqueness of the mapping F, assume that there exists another quadratic mapping $G : X \times X \to Y$ which satisfies (1.1) and (3.3). For fix $x, y \in X$, we know that $H(2^n x, 2^n y) = 4^n H(x,y)$ and $G(2^n x, 2^n y) = 4^n G(x,y)$ for all $n \in \mathbb{N}$. It follows from (3.5) that

$$\begin{aligned}\mathcal{P}_{\mu,\nu}(H(x,y) - G(x,y), t) &= \mathcal{P}_{\mu,\nu}\Big(\frac{H(2^n x, 2^n y)}{4^n} - \frac{G(2^n x, 2^n y)}{4^n}, t\Big) \\ &\geqslant_{L^*} \mathcal{T}\Big(\mathcal{P}_{\mu,\nu}\Big(\frac{H(2^n x, 2^n y)}{4^n} - \frac{f(2^n x, 2^n y)}{4^n}, \frac{t}{2}\Big), \\ &\qquad \mathcal{P}_{\mu,\nu}\Big(-\frac{G(2^n x, 2^n y)}{4^n} + \frac{f(2^n x, 2^n y)}{4^n}, \frac{t}{2}\Big)\Big) \\ &\geqslant_{L^*} \mathcal{T}^2\Big(\mathcal{P'}_{\mu,\nu}\Big(\varphi(2^n x, 2^n x, 2^n y, 2^n y), \frac{4^n(4-\alpha)t}{2}\Big)\Big) \\ &\geqslant_{L^*} \mathcal{T}^2\Big(\mathcal{P'}_{\mu,\nu}\Big(\varphi(x,x,y,y), \frac{4^n(4-\alpha)t}{2\alpha^n}\Big)\Big)\end{aligned}$$

for all $x, y \in X$, all $n \in \mathbb{N}$ and all $t > 0$, where $\mathcal{T}^2(a) = \mathcal{T}(a, a)$ for all $a \in [0, 1]$. Since $\lim_{n\to\infty} \frac{4^n(4-\alpha)t}{2\alpha^n} = \infty$ for all $t > 0$, we get

$$\lim_{n\to\infty} \mathcal{P}'_{\mu,\nu}\Big(\varphi(x, x, y, y), \frac{4^n(4-\alpha)t}{2\alpha^n}\Big) = 1$$

for all $x, y \in X$ and all $t > 0$. Therefore $\mathcal{P}'_{\mu,\nu}(H(x, y) - G(x, y), t) = 1$. Thus it is concluded that $H(x, y) = G(x, y)$. ∎

Corollary 3.2. *Let p be a nonnegative real number with $p < 2$, X be a normed space with norm $\|.\|$, $(Z, \mathcal{P}'_{\mu,\nu}, \mathcal{T})$ be an intuitionistic fuzzy normed space, $(Y, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ be complete intuitionistic fuzzy normed space, and let $z_0 \in Z$. If $f : X \to Y$ is a mapping such that*

$$\begin{aligned}\mathcal{P}_{\mu,\nu}(f([xyz], w^3) &- [f(x, y)f(y, w)f(z, w)], t) + \mathcal{P}_{\mu,\nu}(f(x^3, [yzw]) \\ &- [f(x, y)f(x, z)f(x, w)], t) \\ &\geqslant_{L^*} \mathcal{P}'_{\mu,\nu}((\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)z_0, t)\end{aligned}$$

and

$$\mathcal{P}_{\mu,\nu}(D_q f(x, y), t) \geqslant_{L^*} \mathcal{P}'_{\mu,\nu}((\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)z_0, t)$$

for all $x, y, z, w \in X$ and $t > 0$, then there exists a unique bi-homomorphism $H : X\times X \to Y$ such that

$$\mathcal{P}_{\mu,\nu}(H(x, y) - f(x, y), t) \geqslant_{L^*} \mathcal{P}'_{\mu,\nu}\Big((\|x\|^p + \|y\|^p)z_0, \frac{(4-2^p)t}{2}\Big)$$

for all $x, y \in X$ and $t > 0$.

Theorem 3.3. *Let X be a linear space and let $(Z, \mathcal{P}'_{\mu,\nu}, \mathcal{T}')$ be an IFN-space. Let $\varphi : X^4 \to Z$ be a function, for some real number with $\alpha > 4$, such that*

$$\mathcal{P}'_{\mu,\nu}(\varphi(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}, \frac{w}{2}), t) \geqslant_{L^*} \mathcal{P}'_{\mu,\nu}(\varphi(x, y, z, w), \alpha t) \tag{3.17}$$

for all $x, y, z, w \in X$ and all $t > 0$. Let $(Y, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ be an intuitionistic fuzzy Banach space and let $f : X \times X \to Y$, with $f(0, 0) = 0$, be a mapping such that satisfies in (3.3) and (3.4). Then there exists a unique intuitionistic fuzzy bi-homomorphism $H : X \times X \to Y$ satisfying (1.1) such that

$$\mathcal{P}_{\mu,\nu}\Big(H(x, y) - f(x, y), t\Big) \geqslant_{L^*} \mathcal{P}'_{\mu,\nu}(\varphi(x, x, y, y), (\alpha - 4)t), \tag{3.18}$$

for all $x, y \in X$ and all $t > 0$.

Proof. It follows from (3.6) that

$$\mathcal{P}_{\mu,\nu}(f(x, y) - 4f(\frac{x}{2}, \frac{y}{2}), t) \geqslant_{L^*} \mathcal{P}'_{\mu,\nu}((\varphi(x, x, y, y), t)$$

for all $x, y \in X$ and all $t > 0$.

Consider the set $\Omega := \{g : X \times X \to Y\}$ and introduce the generalized metric on X,

$$\begin{aligned}d(g, h) = \inf\{K \in \mathbb{R}^+ : \mathcal{P}_{\mu,\nu}(g(x, y) - h(x, y), Kt) \geqslant_{L^*} \mathcal{P}'_{\mu,\nu}(\varphi(x, x, y, y), t), \\ \forall x, y \in X, t > 0\}.\end{aligned}$$

Now we consider the linear mapping $J : X \to X$, such that

$$Jg(x, y) := 4g(\frac{x}{2}, \frac{y}{2})$$

for all $x, y \in X$. We prove that J is a strictly contractive mapping of Ω with the Lipschitz constant $\frac{4}{\alpha}$.

Let $g, h \in \Omega$ be given such that $d(g,h) = \varepsilon$. Hence, from (3.9) and (3.17), we have

$$\mathcal{P}_{\mu,\nu}(Jg(x,y) - Jh(x,y), \varepsilon t) = \mathcal{P}_{\mu,\nu}\Big(g\Big(\frac{x}{2}, \frac{y}{2}\Big), h\Big(\frac{x}{2}, \frac{y}{2}\Big), \frac{\varepsilon}{4}t\Big)$$
$$\geqslant_{L^*} \mathcal{P}'_{\mu,\nu}\Big(\varphi\Big(\frac{x}{2}, \frac{x}{2}, \frac{y}{2}, \frac{y}{2}\Big), \frac{1}{4}t\Big) \geqslant_{L^*} \mathcal{P}'_{\mu,\nu}\Big(\varphi(x,x,y,y), \frac{\alpha}{4}t\Big)$$

for all $x, y \in X$ and all $t > 0$. By definition, $d(Jg, Jh) \leqslant \frac{\alpha}{4}\varepsilon$. Therefore,

$$d(Jg, Jh) \leqslant \frac{4}{\alpha} d(g,h)$$

for all $x, y \in X$ and all $t > 0$. It follows from (3.6) that $d(f, Jf) = \frac{1}{\alpha}$. Therefore, by theorem 2.7, J has a unique fixed point in the set $\Delta = \{g \in \Omega : d(g,f) < \infty\}$. Let H be the fixed point of J, that is

$$4H\Big(\frac{x}{2}, \frac{y}{2}\Big) = H(x,y)$$

for all $x, y \in X$, satisfying there exists $K > 0$, such that

$$\mathcal{P}_{\mu,\nu}(f(x,y) - H(x,y), Kt) \geqslant_{L^*} \mathcal{P}'_{\mu,\nu}(\varphi(x,x,y,y), t)$$

for all $x, y \in X$ and all $t > 0$. On the other hand, we have

$$\lim_{n\to\infty} d(J^n f, H) = 0.$$

This implies the following quality

$$H(x,y) := \lim_{n\to\infty} J^n f(x,y) = \lim_{n\to\infty} 4^n f\Big(\frac{x}{2^n}, \frac{y}{2^n}\Big)$$

for all $x, y \in X$ and all $t > 0$. It follows from $d(f,H) \leqslant \frac{1}{1-L} d(f, Jf)$ that

$$d(f,H) \leqslant \frac{1}{\alpha(1-L)}.$$

This implies inequality (3.18). The remainder of proof is similar to the proof of Theorem 3.1. ∎

Corollary 3.4. *Let p be a nonnegative real number with $p > 2$, X be a normed space with norm $\|.\|$, $(Z, \mathcal{P}'_{\mu,\nu}, \mathcal{T})$ be an intuitionistic fuzzy normed space, $(Y, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ be complete intuitionistic fuzzy normed space, and let $z_0 \in Z$. If $f : X \to Y$ is a mapping such that*

$$\mathcal{P}_{\mu,\nu}(f([xyz], w^3) - [f(x,y)f(y,w)f(z,w)], t) + \mathcal{P}_{\mu,\nu}(f(x^3, [yzw])$$
$$- [f(x,y)f(x,z)f(x,w)], t)$$
$$\geqslant_{L^*} \mathcal{P}'_{\mu,\nu}((\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)z_0, t)$$

and

$$\mathcal{P}_{\mu,\nu}(D_q f(x,y), t) \geqslant_{L^*} \mathcal{P}'_{\mu,\nu}((\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)z_0, t)$$

for all $x, y, z, w \in X$ and $t > 0$, then there exists a unique bi-homomorphism $H : X \times X \to Y$ such that

$$\mathcal{P}_{\mu,\nu}(H(x,y) - f(x,y), t) \geqslant_{L^*} \mathcal{P}'_{\mu,\nu}\Big((\|x\|^p + \|y\|^p)z_0, \frac{(2^p - 4)t}{2}\Big)$$

for all $x, y \in X$ and $t > 0$.

4. Intuitionistic Fuzzy Bi-Derivations Stability of Quadratic Functional Equation

In this section, we investigate generalized Hyers-Ulam stability of bi-derivations in IFN-space for the functional equation (1.1).

Theorem 4.1. *Let X be a linear space and let $(Z, \mathcal{P}'_{\mu,\nu}, \mathcal{T}')$ be an IFN-space. Let $f : X \times X \to Y$, with $f(0,0) = 0$, be a mapping for which there exists a mapping $\varphi : X^4 \to Z$ such that, for some $0 < \alpha < 4$ satisfying (3.1), (3.2), (3.4) and*

$$\begin{aligned}&\mathcal{P}_{\mu,\nu}(f([xyz], w) - [f(x, w)yz] - [xf(y, w)z] - [xyf(z, w)], t)\\&\qquad + (f(x, [yzw]) - [f(x, y)zw] - [yf(x, z)w][yzf(x, w)], t)\\&\qquad \geqslant_{L^*} \mathcal{P}'_{\mu,\nu}(\varphi(x, y, z, w), t)\end{aligned} \tag{4.1}$$

for all $x, y, z, w \in X$ and all $t > 0$. Then there exists a unique intuitionistic fuzzy bi-derivation $\delta : X \times X \to Y$ satisfying (1.1) such that

$$\mathcal{P}_{\mu,\nu}\Big(\delta(x, y) - f(x, y), t\Big) \geqslant_{L^*} \mathcal{P}'_{\mu,\nu}(\varphi(x, x, y, y), (4 - \alpha)t), \tag{4.2}$$

for all $x, y \in X$ and all $t > 0$.

Proof. By the same argument as in the proof of the Theorem 3.1, there exists a unique quadratic mapping $\delta : X \times X \to Y$ satisfying (4.2). The mapping δ is given by

$$\delta = \lim_{n\to\infty} \frac{1}{4} f(2^n x, 2^n y)$$

for all $x, y \in X$. It follows from (4.1)

$$\begin{aligned}&\mathcal{P}_{\mu,\nu}(\delta([xyz], w) - [\delta(x, w)yz] - [x\delta(y, w)z] - [xy\delta(z, w)], t)\\&\qquad + \mathcal{P}_{\mu,\nu}(\delta(x, [yzw]) - [\delta(x, y)zw] - [y\delta(x, z)w] - [yz\delta(x, w)], t)\\&= \mathcal{P}_{\mu,\nu}\Big(\frac{1}{4^{3n}} f(2^{3n}[xyz], 2^{3n}w) - \Big[\frac{1}{4^n} f(2^n x, 2^n w)yz\Big]\\&\qquad - \Big[x\frac{1}{4^n} f(2^n y, 2^n w)z\Big] - \Big[xy\frac{1}{4^n} f(2^n z, 2^n w)\Big], t\Big)\\&\qquad + \mathcal{P}_{\mu,\nu}\Big(\frac{1}{4^{3n}} f(2^{3n}x, 2^{3n}[yzw]) - \Big[\frac{1}{4^n} f(2^n x, 2^n y)zw\Big]\\&\qquad - \Big[y\frac{1}{4^n} f(2^n x, 2^n z)w\Big] - \Big[yz\frac{1}{4^n} f(2^n x, 2^n w)\Big], t\Big)\\&= \mathcal{P}_{\mu,\nu}\Big(\frac{1}{4^{3n}} f([2^n x 2^n y 2^n z], 2^{3n}w) - \frac{1}{4^{3n}}[f(2^n x, 2^{3n}w)2^n y 2^n z]\\&\qquad - \frac{1}{4^{3n}}[2^n x f(2^n y, 2^{3n}w)2^n z] - \frac{1}{4^{3n}}[2^n x 2^n y f(2^n z, 2^{3n}w)], t\Big)\\&\qquad + \mathcal{P}_{\mu,\nu}\Big(\frac{1}{4^{3n}} f(2^{3n}x, [2^n y 2^n z 2^n w]) - \frac{1}{4^{3n}}[f(2^{3n}x, 2^n y)2^n z 2^n w]\\&\qquad - \frac{1}{4^{3n}}[2^n y f(2^{3n}x, 2^n z)2^n w] - \frac{1}{4^{3n}}[2^n y 2^n z f(2^{3n}x, 2^n w)], t\Big)\\&\geqslant_{L^*} \mathcal{P}'_{\mu,\nu}(\varphi(2^n x, 2^n y, 2^n z, 2^{3n}w), 4^{3n}t)\\&\qquad + \mathcal{P}'_{\mu,\nu}(\varphi(2^{3n}x, 2^n y, 2^n z, 2^n w), 4^{3n}t))\\&\geqslant_{L^*} 2\mathcal{P}'_{\mu,\nu}\Big(\varphi(x, y, z, w), \frac{4^{3n}t}{\alpha^{3n}}\Big) \to 1\end{aligned}$$

as $n \to \infty$ for all $x, y, z, w \in \mathcal{A}$, because of $\lim_{n\to\infty} \frac{4^{3n}t}{\alpha^{3n}} = \infty$. Thus

$$\delta([xyz], w) = [\delta(x, w)yz] + [x\delta(y, w)z] + [xy\delta(z, w)],$$
$$\delta(x, [yzw]) = [\delta(x, y)zw] + [y\delta(x, z)w] + [yz\delta(x, w)]$$

for all $x, y, z, w \in \mathcal{A}$. Therefore, we conclude that δ is unique $\mathbb{C}$-bilinear satisfying (4.2). ■

Corollary 4.2. *Let p be a nonnegative real number with $p < 2$, X be a normed space with norm $\|.\|$, $(Z, \mathcal{P}'_{\mu,\nu}, \mathcal{T})$ be an intuitionistic fuzzy normed space, $(Y, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ be complete intuitionistic fuzzy normed space, and let $z_0 \in Z$. If $f : X \to Y$ is a mapping such that*

$$\begin{aligned}\mathcal{P}_{\mu,\nu}(f([xyz], w) - [f(x, w)yz] - [xf(y, w)z] - [xyf(z, w)], t)\\ + (f(x, [yzw]) - [f(x, y)zw] - [yf(x, z)w][yzf(x, w)], t)\\ \geqslant_{L^*} \mathcal{P}'_{\mu,\nu}((\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)z_0, t)\end{aligned}$$

and

$$\mathcal{P}_{\mu,\nu}(D_q f(x, y), t) \geqslant_{L^*} \mathcal{P}'_{\mu,\nu}((\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)z_0, t)$$

for all $x, y, z, w \in X$ and $t > 0$, then there exists a unique bi-derivation $\delta : X \times X \to Y$ such that

$$\mathcal{P}_{\mu,\nu}(\delta(x, y) - f(x, y), t) \geqslant_{L^*} \mathcal{P}'_{\mu,\nu}\Big((\|x\|^p + \|y\|^p)z_0, \frac{(4 - 2^p)t}{2}\Big)$$

for all $x, y \in X$ and $t > 0$.

Example 4.3. Let X be a Hilbert space with inner product $\langle ., .\rangle$ and Z be a normed spaced. Denote by $(\mathcal{P}_{\mu,\nu}, \mathcal{T})$ and $(\mathcal{P}'_{\mu,\nu}, \mathcal{T})$ the intuitionistic fuzzy norms given as in Example 2.9 on X and Z, respectively. Let $\|.\|$ be induced norm on X by the inner product $\langle ., .\rangle$ on X. Let $\varphi : X^4 \to Z$ be a mapping defined by

$$\varphi(x, y, z, w) = 2(\|x\| + \|y\| + \|z\| + \|w\|)z_0$$

for all $x, y, z, w \in X$, where z_0 is a fixed unit vector in Z. Define a mapping $f : X \times X \to Y$ by

$$f(x, y) := \langle x, y + x_0\rangle x_0$$

for all $x, y \in X$, where x_0 is a fixed unit vector in X. Then

$$\begin{aligned}\mu(D_q f(x, y), t) &= \mu(-2\langle y, x_0\rangle x_0, t)\\ &= \frac{t}{t + 2|\langle y, x_0\rangle|}\\ &\geqslant \frac{t}{t + 2\|y\|}\\ &\geqslant \frac{t}{t + 2(\|x\| + \|y\| + \|z\| + \|w\|)}\\ &= \mu'(\varphi(x, y, z, w), t)\end{aligned}$$

and

$$\begin{aligned}\nu(D_q f(x,y),t) &= \nu(-2\langle y,x_0\rangle x_0,t)\\ &= \frac{2|\langle y,x_0\rangle|}{t+2|\langle y,x_0\rangle|}\\ &\leqslant \frac{2\|y\|}{t+\|y\|}\\ &\leqslant \frac{2(\|x\|+\|y\|+\|z\|+\|w\|)}{t+2(\|x\|+\|y\|+\|z\|+\|w\|)}\\ &= \nu'(\varphi(x,y,z,w),t)\end{aligned}$$

for all $x,y,z,w \in X$ and all $t>0$. Then it is concluded that

$$\mathcal{P}_{\mu,\nu}(D_q f(x,y),t) \geqslant_{L^*} \mathcal{P}'_{\mu,\nu}(\varphi(x,y,z,w),t).$$

Also we can get

$$\mu'(\varphi(2x,2y,2z,2w),t) = \frac{t}{t+4(\|x\|+\|y\|+\|z\|+\|w\|)} = \mu'(2\varphi(x,y,z,w),t)$$

and

$$\nu'(\varphi(2x,2y,2z,2w),t) = \frac{4(\|x\|+\|y\|+\|z\|+\|w\|)}{t+4(\|x\|+\|y\|+\|z\|+\|w\|)} = \nu'(2\varphi(x,y,z,w),t)$$

for all $x,y,z,w \in X$ and all $t>0$. Therefore

$$\mathcal{P}'_{\mu,\nu}(\varphi(2x,2y,2z,2w),t) = \mathcal{P}'_{\mu,\nu}(2\varphi(x,y,z,w),t)$$

for all $x,y,z,w \in X$ and all $t>0$. Hence the assumptions of Theorem 3.3 (also Theorem 2.7) for $\alpha = 2$ are fulfilled. Therefore, there exist a unique bi-derivation (also bi-homomorphism) $\delta : X \times X \to Y$ such that

$$\mathcal{P}_{\mu,\nu}(\delta(x,y)-f(x,y),t) \geqslant \mathcal{P}'_{\mu,\nu}(4(\|x\|+\|y\|)z_0,2t)$$

for all $x,y \in X$ and all $t>0$.

Theorem 4.4. *Let X be a linear space and let $(Z<\mu',\nu,)$ be an IFNS. Let $f : X\times X \to Y$, with $f(0,0)=0$, be a mapping for which there exists a mapping $\varphi : X^4 \to Z$ such that, for some $\alpha > 4$ satisfying (3.1), (3.4), (3.17) and (4.1). Then there exists a unique bi-derivation $\delta : X \times X \to Y$ such that*

$$\mathcal{P}_{\mu,\nu}(\delta(x,y)-f(x,y),t) \geqslant_{L^*} \mathcal{P}'_{\mu,\nu}(\varphi(x,x,y,y),(\alpha-4)t),$$

for all $x,y \in X$ and all $t>0$.

Proof. The proof is similar to the proof of Theorems 3.3 and 4.1. ∎

Corollary 4.5. *Let p be a nonnegative real number with $p>2$, X be a normed space with norm $\|.\|$, $(Z,\mathcal{P}'_{\mu,\nu},\mathcal{T})$ be an intuitionistic fuzzy normed space, $(Y,\mathcal{P}_{\mu,\nu},\mathcal{T})$ be complete intuitionistic fuzzy normed space, and let $z_0 \in Z$. If $f : X \to Y$ is a mapping such that*

$$\begin{aligned}&\mathcal{P}_{\mu,\nu}(f([xyz],w)-[f(x,w)yz]-[xf(y,w)z]-[xyf(z,w)],t)\\ &\qquad +(f(x,[yzw])-[f(x,y)zw]-[yf(x,z)w][yzf(x,w)],t)\\ &\quad \geqslant_{L^*} \mathcal{P}'_{\mu,\nu}((\|x\|^p+\|y\|^p+\|z\|^p+\|w\|^p)z_0,t)\end{aligned}$$

and

$$\mathcal{P}_{\mu,\nu}(D_q f(x,y),t) \geqslant_{L^*} \mathcal{P}'_{\mu,\nu}((\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)z_0, t)$$

for all $x, y, z, w \in X$ and $t > 0$, then there exists a unique bi-derivation $\delta : X \times X \to Y$ such that

$$\mathcal{P}_{\mu,\nu}(\delta(x,y) - f(x,y), t) \geqslant_{L^*} \mathcal{P}'_{\mu,\nu}\Big((\|x\|^p + \|y\|^p)z_0, \frac{(2^p - 4)t}{2}\Big)$$

for all $x, y \in X$ and $t > 0$.

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