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UNIVALENCY AND CONVOLUTION RESULTS ASSOCIATED TO STARLIKE FUNCTIONS WITH RESPECT TO SYMMETRIC POINTS

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Abstract

In the present paper, the authors introduce some new subclasses of analytic functions in the open unit disc and investigate their inclusion relationships and convolution properties.
Key words: starlike functions with symmetric points, univalent functions, hypergeometric functions, convolution
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1. Introduction and preliminaries. Let $\mathcal{A}$ denote the class of all analytic functions f in the unit disc $\Delta = \{z : z \in \mathbb{C}, |z| < 1\}$ and normalized by $f(0) = f'(0) - 1 = 0$.

A function $f \in \mathcal{A}$, is said to belong to the class S if f is univalent in Δ. Let $S^*(\alpha)$ denote the famous family of starlike functions of order α, and $S_s^*(\alpha)$, $\alpha < 1$, be the family of starlike functions with respect to symmetric points, i.e. the analytic condition

$$\operatorname{Re}\left\{\frac{2zf'(z)}{f(z) - f(-z)}\right\} > \alpha,$$

is satisfied for $z \in \Delta$. Set $S_s^* := S_s^*(0)$. This class was introduced by Sakaguchi [9] and various subclasses of it investigated by many authors such as (for example see [2, 8–11]).

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A function $f \in \mathcal{A}$ is said to be strongly starlike with respect to symmetric points of order $\alpha, 0 < \alpha \leq 1$, if and only if f satisfies the analytic condition

$$\frac{2zf'(z)}{f(z)-f(-z)} \prec \left(\frac{1+z}{1-z}\right)^{\alpha}, \quad z \in \Delta,$$

where $\prec$ denotes the usual subordination. The class of all functions which are strongly starlike with respect to symmetric points of order α is denoted by $S_{s\alpha}$. Clearly $S_{s1} = S_s^*$.

Furthermore for $0 < \lambda \leq 1$ let

$$U(\lambda) = \left\{ f \in \mathcal{A} : \left| \left(\frac{z}{f(z)}\right)^2 f'(z) - 1 \right| < \lambda \right\}.$$

This class is very famous and important in univalent functions theory and many interesting results associated to this class and relevant subclasses of it have been obtained by many authors such as [1,4–7].

This class appeared to be useful in the study of starlike functions with respect to symmetric points. Hence in this paper we aim to define analogue of $U(\lambda)$ for starlike functions with respect to symmetric points and investigate relationship of it and other subclasses of univalent functions.

For $0 < \mu \leq \alpha \leq 1$ the other interesting class of functions is the class

$$SU(\alpha,\mu,\lambda) =$$
$$\left\{ f \in \mathcal{A} : \left| (1-\alpha)\left(\frac{2z}{f(z)-f(-z)}\right)^{\mu} + \alpha \left(\frac{2z}{f(z)-f(-z)}\right)^{\mu+1} f'(z) - 1 \right| < \lambda \right\}$$

for which $f(z) - f(-z) \neq 0$ for $z \in \Delta \backslash \{0\}$.

Note that for $\mu = 1, \alpha = 1$ and for odd function $f \in U(\lambda)$ we have $f \in SU(1,1,\lambda)$.

Finally let the class $R_\alpha, 0 < \alpha \leq 1$, denote the set of all functions in $\mathcal{A}$ such that

$$f'(z) \prec \left(\frac{1+z}{1-z}\right)^{\alpha}, \quad z \in \Delta.$$

Our main result is motivated by this problem. Find conditions on the λ, α for which $f \in SU(\alpha,\mu,\lambda)$ implies that f belongs to $S_{s\alpha}$ or R_α and other relevant subclasses of it.

2. Discussion on $SU(\alpha,\mu,\lambda)$. Suppose that $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ belongs to $SU(\alpha,\mu,\lambda)$ and let

$$g(z) = \frac{f(z)-f(-z)}{2}.$$

Then we have

$$g'(z) = \frac{f'(z) + f'(-z)}{2}$$

and so we can write

$$\left|(1-\alpha)\left(\frac{z}{g(z)}\right)^{\mu} + \alpha\left(\frac{z}{g(z)}\right)^{\mu+1} g'(z) - 1\right|$$

$$\leq \frac{1}{2}\left|(1-\alpha)\left(\frac{2z}{f(z)+f(-z)}\right)^{\mu} + \alpha\left(\frac{2z}{f(z)+f(-z)}\right)^{\mu+1} f'(z) - 1\right|$$

$$+ \frac{1}{2}\left|(1-\alpha)\left(\frac{2z}{f(z)+f(-z)}\right)^{\mu} + \alpha\left(\frac{2z}{f(z)+f(-z)}\right)^{\mu+1} f'(-z) - 1\right| < \lambda.$$

On the other hand

$$(1-\alpha)\left(\frac{z}{g(z)}\right)^{\mu} + \alpha\left(\frac{z}{g(z)}\right)^{\mu+1} g'(z) = 1 + (2\alpha-\mu)a_3 z^2 + \ldots = 1 + \lambda\omega(z), \tag{2.1}$$

where $\omega(z)$ is an analytic function with $|\omega(z)| < 1$ and $\omega(0) = \omega'(0) = 0$. Set

$$p(z) = \left(\frac{z}{g(z)}\right)^{\mu}$$

it is easy to check that

$$(1-\alpha)\left(\frac{z}{g(z)}\right)^{\mu} + \alpha\left(\frac{z}{g(z)}\right)^{\mu+1} g'(z) = p(z) - \frac{\alpha}{\mu} z p'(z). \tag{2.2}$$

Now it follows from (2.1) and (2.2)

$$p(z) - \frac{\alpha}{\mu} z p'(z) = 1 + \lambda\omega(z).$$

Solving this differential equation leads to

$$p(z) = 1 - \frac{\mu\lambda}{\alpha}\int_0^1 \frac{\omega(tz)}{t^{\frac{\mu}{\alpha}+1}} dt. \tag{2.3}$$

Since $\omega'(0) = 0$, with $|\omega(z)| < 1$, we find that

$$|p(z) - 1| \leq \frac{\mu\lambda}{2\alpha-\mu}|z|^2, \quad z \in \Delta, \tag{2.4}$$

and

$$1-\frac{\mu\lambda}{2\alpha-\mu}|z|^2 \leq Rep(z) \leq 1+\frac{\mu\lambda}{2\alpha-\mu}|z|^2. \tag{2.5}$$

By (2.3) we note that the equality holds in each of the last two inequalities (2.4) and (2.5) for the function

$$f(z)=\frac{z}{1+\frac{\mu\lambda}{2\alpha-\mu}z^2}.$$

Thus, for $f \in SU(\alpha,\mu,\lambda)$ we have

$$\mathrm{Re}\left(\frac{2z}{f(z)-f(-z)}\right)^{\mu} > 0 \quad z \in \Delta.$$

On the other hand, (2.4) is equivalent to

$$\left|\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu}-\frac{(2\alpha-\mu)^2}{(2\alpha-\mu)^2-\mu^2\lambda^2}\right| \leq \frac{\mu\lambda(2\alpha-\mu)}{(2\alpha-\mu)^2-\mu^2\lambda^2},$$

which gives

$$\mathrm{Re}\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu} \geq \frac{2\alpha-\mu}{2\alpha-\mu+\lambda\mu}.$$

3. Main results. We start with the following important theorem.

Theorem 3.1. I) *Let* $f \in SU(1,\mu,\lambda)$ *and* $\gamma \in (0,1]$. *Define*

$$\lambda^* = \left[\frac{(2-\mu)^2\sin^2\left(\frac{\pi\gamma}{2}\right)\left[(\mu-1)^2+1-\mu(2-\mu)\cos\left(\frac{\pi\gamma}{2}\right)\right]}{8(\mu-1)^2+2\mu^2(2-\mu)^2\sin^2\left(\frac{\pi\gamma}{2}\right)}\right]^{1/2} \tag{3.1}$$

then $f \in S_{s\gamma}$ *for* $0<\lambda<\lambda^*$.

II) $f \in SU(1,1,\lambda) \Rightarrow f \in R_\gamma$ *for* $0<\lambda<\sin\left(\frac{\pi\gamma}{6}\right)$.

Proof. Suppose that $f \in SU(1,\mu,\lambda)$ for some $0<\mu\leq 1$ and λ in $(0,1]$. Then, by definition of $SU(1,\mu,\lambda)$ and (2.4) we have

$$\left|\left(\frac{2z}{f(z)-f(-z)}\right)^{\mu+1} f'(z)-1\right| < \lambda, \quad z \in \Delta, \tag{3.2}$$

and

$$\left|\left(\frac{2z}{f(z)-f(-z)}\right)^{\mu}-1\right| < \frac{\mu\lambda}{2-\mu}, \quad z \in \Delta. \tag{3.3}$$

Therefore it follows that

$$\left|\arg\left(\frac{2z}{f(z)-f(-z)}\right)^{\mu+1} f'(z)\right| \le \arcsin(\lambda), \tag{3.4}$$

and

$$\left|\arg\left(\frac{2z}{f(z)-f(-z)}\right)^{\mu}\right| < \arcsin\left(\frac{\mu\lambda}{2-\mu}\right). \tag{3.5}$$

Using (3.4) and (3.5) and the addition formula for the inverse of sin function, namely,

$$\arcsin(x) + \arcsin(y) = \arcsin\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$$

we find that

$$\begin{aligned} &\left|\arg\left(\frac{2z}{f(z)-f(-z)}\right) f'(z)\right| \\ &\le \left|\arg\left(\frac{2z}{f(z)-f(-z)}\right)^{\mu+1} f'(z)\right| + \left|\arg\left(\frac{2z}{f(z)-f(-z)}\right)^{\mu}\right| \\ &< \arcsin(\lambda) + \arcsin\left(\frac{\mu\lambda}{2-\mu}\right) \\ &= \arcsin\left[\lambda\sqrt{1-\frac{\mu^2\lambda^2}{(2-\mu)^2}} + \frac{\mu\lambda}{2-\mu}\sqrt{1-\lambda^2}\right]. \end{aligned}$$

Thus, $f \in S_{s\gamma}$, whenever $\lambda \in (0, \lambda^*]$. Here λ^* is the solution of the equation

$$\begin{aligned} &\left[16(1-\mu)^2 + 4\mu^2(2-\mu)^2\sin^2\left(\frac{\pi\gamma}{2}\right)\right]t^2 \\ &-4t(\mu^2-2\mu+2)(2-\mu)^2\sin^2\left(\frac{\pi\gamma}{2}\right) \\ &+(2-\mu)^4\sin^4\left(\frac{\pi\gamma}{2}\right) = 0, \end{aligned}$$

which proves (I).

For the proof of the second part, we recall that

$$|\arg f'(z)| \le \left|\arg\left(\frac{2z}{f(z)-f(-z)}\right)^2 f'(z)\right| + 2\left|\arg\left(\frac{2z}{f(z)-f(-z)}\right)\right|$$

$$\leq \arcsin(\lambda) + 2\arcsin(\lambda) = 3\arcsin(\lambda),$$

which proves (II). $\square$

Theorem 3.2. I) *Let $f \in SU(1,\mu,\lambda)$ and λ^* be as Theorem 3.1. Then for $\lambda^* \leq \lambda \leq 1$, $f \in S_{s\gamma}$ in $|z| < r^*$, where r^* is defined by*

$$r^* = \left(\frac{(\lambda^2 + M^2)\sin^2\left(\frac{\pi\gamma}{2}\right) - 2\lambda M \sin^2\left(\frac{\pi\gamma}{2}\right)\cos\left(\frac{\pi\gamma}{2}\right)}{(\lambda^2 - M^2)^2 + 4\lambda^2 M^2 \sin^2\left(\frac{\pi\gamma}{2}\right)} \right)^{1/4}, \tag{3.6}$$

with $M = \dfrac{\mu\lambda}{2-\mu}$.

II) *Let $f \in SU(1,1,\lambda)$. For $\sin\left(\frac{\pi\gamma}{6}\right) \leq \lambda \leq 1$, $f \in R_\gamma$ in $|z| < r$, where $\gamma \in (0,1]$, and, $0 < r < \sqrt{\dfrac{\sin(\pi\gamma/6)}{\lambda}}$.*

Proof. Let $f \in SU(1,\mu,\lambda)$ for some $0 < \mu \leq 1$ and λ in $(0,1]$. Following the proof of Theorem 3.1, we obtain

$$\left| \arg \left(\frac{2z}{f(z) - f(-z)} \right)^{\mu+1} f'(z) \right| \leq \arcsin(\lambda r^2), \tag{3.7}$$

and

$$\left| \arg \left(\frac{2z}{f(z) - f(-z)} \right)^{\mu} \right| < \arcsin\left(\frac{\mu\lambda}{2-\mu} r^2 \right). \tag{3.8}$$

Combining the last two inequalities, we get

$$\left| \arg \left(\frac{2z}{f(z) - f(-z)} \right) f'(z) \right| < \arcsin\left[\lambda r^2 \sqrt{1 - \frac{\mu^2\lambda^2}{(2-\mu)^2} r^2} + \frac{\mu\lambda}{2-\mu} r^2 \sqrt{1 - \lambda^2 r^2} \right].$$

By some calculation, we see that the right hand side of the last inequality is less than or equal to $\dfrac{\pi\gamma}{2}$ provided that $0 < r < r^*$, where r^* is defined by (3.6).

For the proof of the second part, we recall that

$$|\arg f'(z)| \leq \left| \arg \left(\frac{2z}{f(z) - f(-z)} \right)^2 f'(z) \right| + 2 \left| \arg \left(\frac{2z}{f(z) - f(-z)} \right) \right|$$

$$\leq \arcsin(\lambda r^2) + 2\arcsin(\lambda r^2),$$

which proves (II). $\square$

Corollary 3.1. *Let $f \in SU(1,1,\lambda)$ and $\gamma \in (0,1]$. Then we have the following*

(i) $f \in S_{s\gamma}$ *for,* $0 < \lambda < \sin\left(\frac{\pi\gamma}{4}\right)$. *In particular,* $f \in S_s^*$ *whenever* $0 < \lambda \le \frac{1}{\sqrt{2}}$.

(ii) $f \in S_{s\gamma}$ *in* $|z| < \sqrt{\frac{\sin(\pi\gamma/4)}{\lambda}}$ *if,* $\sin\left(\frac{\pi\gamma}{4}\right) < \lambda \le 1$. *In particular,* $f \in S_s^*$ *in* $|z| < \frac{1}{\sqrt{\sqrt{2}\lambda}}$ *whenever* $\frac{1}{\sqrt{2}} < \lambda \le 1$.

(iii) $f \in R_\gamma$ *for,* $0 < \lambda < \sin\left(\frac{\pi\gamma}{6}\right)$. *In particular,* $f \in R_1$ *whenever* $0 < \lambda \le \frac{1}{2}$.

(iv) $f \in R_\gamma$ *in* $|z| < \sqrt{\frac{\sin(\pi\gamma/6)}{\lambda}}$ *if,* $\sin\left(\frac{\pi\gamma}{6}\right) < \lambda \le 1$. *In particular,* $f \in R_1$ *in* $|z| < \frac{1}{\sqrt{2\lambda}}$ *whenever* $\frac{1}{2} < \lambda \le 1$.

Theorem 3.3. *Let $0 < \mu \le \alpha \le 1$ and $\lambda \in (0,1]$. If $f \in SU(\alpha,\mu,\lambda)$, then for $0 < \beta < 1$, we have*

$$\left|\frac{2zf'(z)}{f(z)-f(-z)} - \frac{1}{2\beta}\right| < \frac{1}{2\beta}, \quad for \quad |z| < r_0,$$

where

$$1)\ r_0 = \sqrt{\frac{\beta}{\beta\lambda + \left(\frac{\mu\lambda}{2\alpha} - \mu\right)}} \quad if \quad 0 < \beta \le \frac{1}{2},$$

$$2)\ r_0 = \sqrt{\frac{1-\beta}{\beta\lambda + \left(\frac{\mu\lambda}{2\alpha} - \mu\right)}} \quad if \quad \frac{1}{2} < \beta < 1.$$

Proof. Let $f \in SU(\alpha,\mu,\lambda)$. Then, by representation (2.1) and (2.3), it follows that

$$\frac{2z}{f(z)-f(-z)}f'(z) = (1+\lambda\omega(z))\left[1 - \frac{\mu\lambda}{\alpha}\int_0^1 \frac{\omega(tz)}{t^{\frac{\mu}{\alpha}+1}}dt\right]^{-1}, \tag{3.9}$$

where ω is Schwarz function with $\omega(0) = \omega'(0) = 0$. By Schwarz lemma we obtain that $|\omega(z)| \le |z|^2$ for $z \in \Delta$ and therefore, an application of the triangle inequality implies that

$$\left|\frac{2z}{f(z)-f(-z)}f'(z) - \frac{1}{2\beta}\right|$$

$$= \frac{1}{2\beta}\left|2\beta - 1 + 2\beta\lambda\omega(z) + \frac{\mu\lambda}{\alpha}\int_0^1 \frac{\omega(tz)}{t^{\frac{\mu}{\alpha}+1}}dt\right|\left|1 - \frac{\mu\lambda}{\alpha}\int_0^1 \frac{\omega(tz)}{t^{\frac{\mu}{\alpha}+1}}dt\right|^{-1}$$

$$\leq \frac{1}{2\beta}\left[\frac{(2\alpha-\mu)(|2\beta-1| + 2\beta\lambda|z|^2) + \mu\lambda|z|^2}{(2\alpha-\mu) - \mu\lambda|z|^2}\right].$$

Note that the bracketed term in the right hand side of the last inequality is less than 1 if

$$|2\beta - 1| + \left[2\beta\lambda + \frac{2\mu\lambda}{2\alpha-\mu}\right]|z|^2 - 1 \leq 0. \tag{3.10}$$

But it is easy to check that with conditions stated in the hypothesis of Theorem 3.3, (3.10) holds true and we complete the proof. □

Example 3.1. Let $f \in SU(\alpha, \mu, \lambda)$ with $0 < \mu \leq 1$. Then

(i) $\dfrac{2z}{f(z) - f(-z)}f'(z) \prec 1 + z$ whenever $0 < \lambda \leq \dfrac{1}{3}$.

(ii) $\dfrac{2z}{f(z) - f(-z)}f'(z) \prec \dfrac{2(1-z)}{2-z}$ whenever $0 < \lambda \leq \dfrac{1}{7}$.

4. Convolution properties. If f and g are analytic functions on Δ with

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} b_n z^n,$$

then the convolution (Hadamard product) of f and g, is an analytic function on Δ given by

$$(f * g)(z) = \sum_{n=0}^{\infty} b_n a_n z^n, \quad z \in \Delta.$$

For $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in $\mathcal{A}$, we have a natural convolution operator defined by

$$zF(a, b; c; z) * f(z) = \sum_{n=1}^{\infty} \frac{(a, n-1)(b, n-1)}{(c, n-1)(1, n-1)} a_n z^n, \tag{4.1}$$

which is analytic in the unit disc Δ. In (4.1), $(a, 0) = 1$ for $a \neq 0$ and the rising factorial notation

$$(a, n) = a(a+1)(a+2)\cdots(a+n-1), \quad n \geq 1$$

is used. To avoid division by 0, the parameter c in (4.1) should be neither 0 nor a negative integer. Here $F(a, b; c; z)$ denotes the Gauss hypergeometric function which is analytic in Δ. It is well known that for $c, a > 0$ and $c \geq \max(2, a)$, the function $F(1, a; c; z)$ is univalent convex function in Δ.

Theorem 4.1. *Let $f \in SU(1, 1, \lambda)$ and $c \in \mathbb{C}$ with $\operatorname{Re} c \geq 0 \neq c$ in Δ, such that*

$$\left(\frac{2z}{f(z) - f(-z)}\right) * F(1, c; c+1; z^2) \neq 0 \quad z \in \Delta,$$

and G be the transform defined by

$$G(z) = \frac{z}{\left(\frac{z}{f(z) - f(-z)}\right) * F(1, c; c+1; z^2)}.$$

Then we have the following:

(*i*) $G \in SU\left(1, 1, \frac{\lambda|c|}{|c+1|}\right)$,

(*ii*) $G \in S_{s\gamma}$ *whenever* $0 < \lambda \leq \frac{|c| \sin(\pi\gamma/4)}{|c+1|}$,

(*iii*) $G \in R_\gamma$ *whenever* $0 < \lambda \leq \frac{|c| \sin(\pi\gamma/6)}{|c+1|}$.

Proof. From the definition of $G(z)$ we have

$$\frac{z}{G(z)} = \left(\frac{2z}{f(z) - f(-z)}\right) * F(1, c; c+1; z^2). \tag{4.2}$$

Using the series expansion of $F(1, c; c+1, z^2)$, we can write

$$F(1, c; c+1; z^2) = 1 + \sum_{n=1}^{\infty} \frac{(c)_n}{(c+1)_n} z^{2n} = 1 + \sum_{n=1}^{\infty} \frac{c}{c+n} z^{2n}, \quad z \in \Delta, \tag{4.3}$$

which yields

$$2cF(1, c; c+1; z^2) + zF'(1, c; c+1; z^2) = \frac{2c}{1 - z^2}, \quad z \in \Delta. \tag{4.4}$$

From (4.2) and (4.4), we obtain

$$2c\frac{z}{G(z)} + z\left(\frac{z}{G(z)}\right)' = \frac{4cz}{f(z) - f(-z)}. \tag{4.5}$$

Now, set

$$p(z) = \left(\frac{z}{G(z)}\right)^2 G'(z),$$

then $p(z)$ is analytic on Δ (with $p(0) = 1$ and $p'(0) = 1$).

Differentiating $\dfrac{z}{G(z)}$ shows that

$$2c\ \frac{z}{G(z)} + z\left(\frac{z}{G(z)}\right)' = (2c+1)\ \frac{z}{G(z)} - \left(\frac{z}{G(z)}\right)^2 G'(z),$$

and so from (4.5) one then obtains that

$$p(z) = (2c+1)\frac{z}{G(z)} - \frac{4cz}{f(z)-f(-z)} \tag{4.6}$$

and

$$zp'(z) = (2c+1)\ z\left(\frac{z}{G(z)}\right)' - 2cz\left(\frac{2z}{f(z)-f(-z)}\right)' \tag{4.7}$$

$$= -2c(2c+1)\frac{z}{G(z)} + 2c(2c+1)\frac{2z}{f(z)-f(-z)} - 2cz\left(\frac{2z}{f(z)-f(-z)}\right)'.$$

We note that $f \in SU(1,1,\lambda)$ implies that the function $H(z)$ defined by

$$H(z) = \frac{f(z)-f(-z)}{2}$$

satisfies

$$\left|\left(\frac{z}{H(z)}\right)^2 H'(z) - 1\right| < \lambda.$$

Combining the relations (4.6) and (4.7) we observe

$$2cp(z) + zp'(z)$$

$$= 2c(2c+1)\frac{z}{G(z)} - 4c^2\frac{z}{H(z)} - 2c(2c+1)\frac{z}{G(z)} + 2c(2c+1)\frac{z}{H(z)} - 2cz\left(\frac{z}{H(z)}\right)'$$

$$= 2c\left[\frac{z}{H(z)} - z\left(\frac{z}{H(z)}\right)'\right]$$

$$= 2c\left(\frac{z}{H(z)}\right)^2 H'(z).$$

Now it follows that

$$\left|p(z) + \frac{1}{2c}zp'(z) - 1\right| < \lambda, \quad z \in \Delta,$$

and so (because $p'(0) = 0$), from the work of HALLENBECK and RUSCHEWEYH [3] we deduce that

$$|p(z) - 1| \leq \frac{\lambda|c|}{|c+1|}|z|^2, \quad z \in \Delta.$$

Hence the conclusion (i) follows and the second and third parts are a consequence of Theorem 3.1. □

REFERENCES

[1] AGHALARY R., A. EBADIAN (2009) Univalence of certain linear operator defined by hypergeometric function, J. Inequal. Appl., **2009**, 1–12.

[2] GAO C.-Y., S.-Q. ZHOU (2005) On a class of analytic functions related to the starlike functions, Kyungpook Math. J., **45**, 123–130.

[3] HALLENBECK D. J., ST. RUSCHEWEYH (1975) Subordination by convex functions, Proc. Amer. Math. Soc., **52**, 191–195.

[4] OBRADOVIC M., S. PONNUSAMY (2007) Univalence and starlikeness of certain transforms defined by convolution of analytic functions, J. Math. Anal. Appl., **337**, 758–767.

[5] OBRADOVIC M., S. PONNUSAMY (2005) Radius properties for subclasses of univalent functions, Analysis (Munich), **25**, 183–188.

[6] PONNUSAMY S., P. VASUNDHRA (2005) Criteria for univalence, starlikeness and convexity, Ann. Polon. Math., **85**, 121–133.

[7] PONNUSAMY S., P. SAHOO (2005) Geometric properties of certain linear integral transforms, Bull. Belg. Math. Soc. Simon Stevin, **12**, 95–108.

[8] REDDY T. R., P. T. REDDY (2007) Neighbourhoods of a certain subclass of $SP(\beta)$, J. Analysis, **15**, 239–245.

[9] SAKAGUCHI K. (1959) On certain univalent mapping, J. Math. Soc. Japan., **11**, 72–75.

[10] SUDHARSAN T. V., P. BALASUBRAHMANYAM, K. G. SUBRAMANIAN (1998) On functions starlike with respect to symmetric and conjugate points, Taiwanese J. Math., **2**, 57–68.

[11] WANG Z.-G., C.-Y. GAO, S.-M. YUAN (2006) On certain new subclass of close-to-convex functions, Mat. Vesnik, **58**(3–4), 119–124.

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