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CONSTRUCTION OF THE DIRAC OPERATOR ON THE q-DEFORMED QUANTUM SPACE $EAdS^2$ USING A GENERALIZED q-DEFORMED GINSPARG–WILSON ALGEBRA

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We construct q-deformed Dirac and chirality operators on the q-deformed quantum space $EAdS^2$ using a generalized quantum Ginsparg–Wilson algebra. We show that in the limit $q \to 1$, these operators become the Dirac and chirality operators on the undeformed quantum space $EAdS^2$.

Keywords: Ginsparg–Wilson algebra, Dirac operator, quantum group

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1. Introduction

In the context of the Connes–Lott approach to noncommutative geometry, Dirac and chirality operators are two important self-adjoint operators. A unital spectral triple [1]–[3] $(\mathcal{A}, \mathcal{H}, D)$ consists of a complex unital $*$-algebra $\mathcal{A}$ faithfully $*$-represented by bounded operators on a separable Hilbert space $\mathcal{H}$ and a self-adjoint operator $D\colon \mathcal{H} \to \mathcal{H}$ (the Dirac operator) with the properties that

- the resolvent $(D-\lambda)^{-1}$, $\lambda \notin \mathbb{R}$ is a compact operator on $\mathcal{H}$ and
- the commutator $[D, \pi(a)]$ is a bounded operator on $\mathcal{H}$ for all $a \in \mathcal{A}$ (here, $\pi(a)$ is the corresponding $a \in \mathcal{A}$ operator in $\mathcal{H}$).

A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is said to be even if there exists a $\mathbb{Z}_2$-grading of $\mathcal{H}$, i.e., an operator $\gamma\colon \mathcal{H} \to \mathcal{H}$ such that $\gamma^* = \gamma$ and $\gamma^2 = 1$ for which $\gamma D + D\gamma = 0$ and $\gamma a = a\gamma$ for all $a \in \mathcal{A}$. Otherwise, the spectral triple is said to be odd. There are no chirality operators for manifolds of odd dimensions, and the Dirac operator in this case describes only the differential structures. There are three types of Dirac and chirality operators on the quantum two-sphere: the Ginsparg–Wilson operator D_{GW} [4]–[8], the Watamura–Watamura operator D_{WW} [9], and the Grosse–Klimcik–Presnajder operator D_{GKP} [10]. These three types of Dirac operators were compared with one another in [11].

The idea of q-deformed geometry was studied extensively in the late 1980s and the 1990s. The q-deformed Hopf fibration was studied in the framework of the Hopf–Galois extension in [12]. Podleś introduced the quantum sphere in [13]. The q-deformed Dirac operator on the Podleś quantum sphere was studied using different approaches [14]–[19]. The q-deformed Watamura–Watamura operator D^q_{WW} was considered in [14], where the Dirac and chirality operators on a noncommutative space were constructed

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using $U_q(su(2))$ as the symmetry group. It was proved that the Dirac operator is covariant and the full rotational invariance of the Dirac operator is recovered in the commutative limit where the underlying space is the Podleś sphere. It was further shown that the Dirac operator reduces to the operator obtained in [9]. The goal in [16] was to describe the q-deformed version of the Grosse–Klimcik–Presnajder operator on the quantum sphere. In addition, there are a few studies using different methods to formulate noncommutative and q-deformed analogues of a noncompact surface with a constant negative curvature. Related to our work, on the quantum AdS^2 space, the Watamura–Watamura operator D^q_{WW} was studied in [20], and the Ginsparg–Wilson operator D_{GW} was studied in [21].

Here, we study the q-deformed quantum Ginsparg–Wilson Dirac operator and the chirality operator using the q-deformed Ginsparg–Wilson algebra on the quantum AdS^2 space [22].

This paper is organized as follows. In Sec. 2, we briefly review the noncompact version of the first Hopf fibration $SU(1,1) \longrightarrow EAdS^2$ with a special focus on projectors and projective modules of the sections of this principal bundle. We also consider the q-deformed quantum version of the noncompact first Hopf fibration and especially study the q-deformed projectors and q-deformed projective modules of the Hopf–Galois extension of the fibration $SU(1,1) \longrightarrow EAdS^2$. In Sec. 3, we consider the spin-1/2 q-deformed pseudoprojectors of the left and right q-deformed projective module. In Sec. 4, we construct the q-deformed quantum Ginsparg–Wilson algebra and also the q-deformed quantum Dirac and chirality operators using the q-deformed left and right projectors and their corresponding idempotents. At each step, we compare our results with the limit as $q \to 1$.

2. Theoretical formalism

We consider the $U(1)$ noncompact principal fibration π with AdS^3 as the total space over the base space $EAdS^2$ (see [23] for the compact case),

$$U(1) \xrightarrow{\text{right } U(1)\text{-action}} AdS^3 \xrightarrow{\pi} EAdS^2, \tag{2.1}$$

where the Euclidean AdS^2 ($EAdS^2$) space is a two-sheet hyperboloid and AdS^2 is a connected one-sheet hyperboloid [24]. The $EAdS^2$ space of radius 1 can be defined as

$$\mathbf{x} \cdot \mathbf{x} = x^i \eta_{ij} x^j = x^2 + y^2 - z^2 = -1. \tag{2.2}$$

The Minkowskian metric $\eta^{ij} = \eta_{ij} = \mathrm{diag}(1, 1, -1)$ raises and lowers the indices. Let $B_{\mathbb{C}} = C^\infty(AdS^3, \mathbb{C})$ and $A_{\mathbb{C}} = C^\infty(EAdS^2, \mathbb{C})$ denote the respective algebras of $\mathbb{C}$-valued smooth functions on the total manifold AdS^3 and the base manifold $EAdS^2$ under pointwise multiplication. The elements of $B_{\mathbb{C}}$ can be classified into the right modules

$$C^\infty_n(AdS^3, \mathbb{C}) = \{\varphi\colon AdS^3 \to \mathbb{C},\ \varphi(p \cdot \omega) = \omega^{-n} \cdot \varphi(p),\ \ p \in AdS^3,\ \omega \in U(1)\}$$

over the pull back of the $A_{\mathbb{C}}$. The left actions of the $U(1)$ group on complex numbers are labeled by an integer n characterizing the bundle.

The Serre–Swan theorem [25] states that for a compact smooth manifold, there is an equivalence between the smooth vector bundles over that manifold and the finitely generated projective modules. It is well known in algebraic K-theory that corresponding to these bundles, there are pseudoprojectors P_n such that for the associated vector bundle

$$E^{(n)} = AdS^3 \times_{U(1)} \mathbb{C} \xrightarrow{\pi} EAdS^2, \tag{2.3}$$

the right $A_{\mathbb{C}}$-module of sections $\Gamma^\infty(EAdS^2, E^{(n)})$ isomorphic to $C^\infty_{(n)}(AdS^3, \mathbb{C})$ is equivalent to the image in the free module $(A_{\mathbb{C}})^{(n+1)} = C^\infty(EAdS^2, \mathbb{C}) \otimes \mathbb{C}^{n+1}$ of a pseudoprojector P_n: $\Gamma^\infty(EAdS^2, E^{(n)}) = P_n(A_{\mathbb{C}})^{n+1}$. The pseudoprojector P_n is a Λ-pseudo Hermitian operator [26] of rank 1 over $\mathbb{C}$,

$$P_n \in M_{n+1}(A_{\mathbb{C}}), \qquad P_n^2 = P_n, \qquad P_n^\dagger = \Lambda P_n \Lambda^{-1}, \qquad \operatorname{Tr} P_n = 1, \tag{2.4}$$

where Λ is a Hermitian, involutory, and unitary matrix. In the case of a compact Hopf fibration such as $S^3 \longrightarrow S^2$, the property of pseudo-Hermiticity reduces to Hermiticity. For the right Λ-pseudo $A_{\mathbb{C}}$-module of sections $\Gamma^\infty(EAdS^2, E^{(n)})$, there exist $n+1$ Λ-pseudoprojectors $P_1, P_2, \ldots, P_{n+1}$ with the same rank 1. Therefore, we can write the Λ-pseudo free module $(A_{\mathbb{C}})^{n+1}$ as a direct sum of the Λ-pseudoprojective $A_{\mathbb{C}}$-modules,

$$(A_{\mathbb{C}})^{n+1} = \bigoplus_{i=1}^{n+1} P_i (A_{\mathbb{C}})^{n+1}. \tag{2.5}$$

A Λ-pseudoprojective module is a projective module constructed from a Λ-pseudoprojector instead of a projector. In Sec. 3, we show how the property of Λ-pseudo-Hermiticity allows studying pseudoprojectors with the spin $s = 1/2$ and expressing the Λ-pseudo free modules in terms of the Λ-pseudoprojective $A_{\mathbb{C}}$-modules on the quantum $EAdS^2$ space. The commutative algebra $A_{\mathbb{C}}$ is replaced with the noncommutative matrix algebra $M_{2\ell+1}(\mathbb{C})$ if we want to pass from the usual to a noncommutative matrix geometry.

A noncommutative geometry contains no points: instead of the coordinates x_i in $EAdS^2$, the angular momentum generators in the unitary irreducible ℓ-representation space play the role of points in the quantum space $EAdS^2$. We let $\mathbf{L}$ denote the angular momentum operator on the commutative space $EAdS^2$ and $\mathcal{A}_\ell = \{\alpha \in M_{2\ell+1}(\mathbb{C})\}$ denote the quantum Hermitian matrix algebra. Any element α can be expressed in terms of the basis elements (with the unitary ℓ-representation of $su(1,1)$ taken into account) of the algebra of the angular momentum $\mathbf{L} = (L_1, L_2, L_3)$ defined as

$$[L_i, L_j] = iC_{ij}{}^k L_k, \tag{2.6}$$

where $C_{ij}{}^k = \eta^{kl} C_{ijl}$, $C_{123} = 1$, and $C_{ij}{}^k$ are totally antisymmetric. The structure constants $C_{ij}{}^k$ satisfy the relation

$$C_{im}{}^k \eta^{ij} C_{jl}{}^n = \eta^n_m \eta^k_l - \eta_{ml}\eta^{kn}. \tag{2.7}$$

The $C(EAdS^2)$-module $\Gamma^\infty(EAdS^2, E^{(n)})$, where $C(EAdS^2)$ is the commutative algebra of functions on $EAdS^2$, is the module of sections in the Hopf fibration $AdS^3 \xrightarrow{U(1)} EAdS^2$. In the quantum case, this algebra is noncommutative, and its left and right modules are therefore not isomorphic. In this case, with each angular momentum operator $\mathbf{L}$, we associate two linear operators $\mathbf{L}^{\mathrm{L}}$ and $\mathbf{L}^{\mathrm{R}}$ with the left and right actions on $\mathcal{A}_\ell$:

$$L_i^{\mathrm{L}}\alpha = L_i\alpha, \qquad L_i^{\mathrm{R}}\alpha = \alpha L_i, \quad \alpha \in \mathcal{A}_\ell. \tag{2.8}$$

The operators L_i^{L} and L_j^{R} form the same $su(1,1)$ algebra with the Casimir operator C:

$$C \equiv [L_i^{\mathrm{L}}, L_j^{\mathrm{L}}] = i\epsilon_{ijk} L_k^{\mathrm{L}}. \tag{2.9}$$

We use $\mathbf{L}^{\mathrm{L}}$ and $\mathbf{L}^{\mathrm{R}}$ to define the quantum version of the orbital momentum operator $\boldsymbol{\mathcal{L}}$ on the quantum space $EAdS^2$. We define $\boldsymbol{\mathcal{L}}$ by the adjoint action of L_i on $\mathcal{A}_\ell$:

$$\mathcal{L}_i\alpha = (L_i^{\mathrm{L}} - L_i^{\mathrm{R}})\alpha = \operatorname{ad}_{L_i}\alpha = [L_i, \alpha]. \tag{2.10}$$

Using the coordinates x_i, we can describe the hypersurface $EAdS^2$ with a constant negative curvature as a surface embedded in three-dimensional flat Minkowski space (2.2). With the Casimir operator of the $su(1,1)$ algebra in mind, we define the parameter μ as

$$\mu = \frac{1}{\sqrt{\ell(1-\ell)}}. \tag{2.11}$$

Relation (2.6) then becomes

$$[x_i, x_j] = \frac{i}{\sqrt{\ell(1-\ell)}} C_{ij}{}^k x_k = i\mu C_{ij}{}^k x_k. \tag{2.12}$$

We obtain the commutative coordinates x_i from the coordinates of the quantum space $EAdS^2$ as a limit case:

$$x_i = \lim_{\ell\to\infty} \frac{L_i^{\mathrm{L,R}}}{\ell}. \tag{2.13}$$

In this case, ℓ^{-1} plays the role of the noncommutative parameter, and we therefore obtain the orbital momentum operator on the commutative space $EAdS^2$ by taking the limit as $\ell \to \infty$,

$$\lim_{l\to\infty} (L_i^{\mathrm{L}} - L_i^{\mathrm{R}}) = -iC_{ij}{}^k x^j \frac{\partial}{\partial x^k}. \tag{2.14}$$

The quantum Hopf bundle is a $U(1)$-bundle over the $EAdS_q^2$ whose total manifold is the manifold of the quantum group $SU_q(1,1)$ (see [27] for the compact case). The space $EAdS_q^2$ is a quantum homogeneous space $SU_q(1,1)$. We let $A(SU_q(1,1))$, $A(EAdS_q^2)$, and $A(U(1))$ denote the respective coordinate algebras of the total space $SU_q(1,1)$, the base space $EAdS_q^2$, and the fiber $U(1)$. The coordinate algebra $A(SU_q(1,1))$ and the quantum universal enveloping algebra $\mathcal{U}_q(su(1,1)) \equiv su_q(1,1)$ are respectively a $*$-algebra and a Hopf $*$-algebra [12].

The algebra inclusion $A(EAdS_q^2) \hookrightarrow A(SU_q(1,1))$, which is a Hopf–Galois extension, is a quantum Hopf bundle. It is the algebraic version of the first quantum noncompact Hopf fibration $SU_q(1,1) \xrightarrow{U_q(1)} EAdS_q^2$. The associated quantum bundle is given by

$$E_q^{(n)} := \{x \in A(SU_q(1,1)) \colon\ k \cdot x = q^{n/2} x,\ k \in A(U_q(1))\}. \tag{2.15}$$

Each $E_q^{(n)}$ is clearly an $A(EAdS_q^2)$-bimodule equivalent to the image in the free module $(A(EAdS_q^2))^{n+1}$ of a self-adjoint quantum projector $p_q^{(n)}$ in $\mathrm{Mat}_{n+1}(A(EAdS_q^2))$ for $n \geq 0$. We identify $E_q^{(n)}$ with the left $A(EAdS_q^2)$-module of sections $(A(EAdS_q^2))^{n+1} p_q^{(n)}$.

We have a q-deformed $SU(1,1)$ Lie algebra, i.e., $U_q(su(1,1)) \equiv su_q(1,1)$, with three generators J_3, J_+, and J_- satisfying the commutation relations

$$[J_+, J_-] = -[J_3], \qquad [J_3, J_\pm] = \pm J_\pm, \tag{2.16}$$

where we use the notation

$$[n] = [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

The structure of the Hopf algebra is given by

$$\begin{aligned} &\Delta(J_\pm) = J_\pm \otimes q^{J_3} + q^{-J_3} \otimes J_\pm, \qquad \Delta(J_3) = J_3 \otimes 1 + 1 \otimes J_3, \\ &S(J_\pm) = -q^{\pm 1} J_\pm, \qquad S(J_3) = -J_3, \qquad \varepsilon(J_\pm) = \varepsilon(J_3) = 0, \end{aligned} \tag{2.17}$$

where Δ, S, and ε are the respective coproduct, antipode, and counit. The generators of $U_q(su(1,1))$ act on the spin-j representation as

$$J_\pm|j,m\rangle = ([m \mp j]_q[m \pm j \pm 1]_q)^{1/2}|j,m\pm 1\rangle, \qquad J_3|j,m\rangle = q^m|j,m\rangle, \tag{2.18}$$

where $j > 0$ and $m = j, j+1, j+2, \dots$. The result of the action of the quantum Casimir operator C_q is given by [19]

$$C_q|j,m\rangle = \frac{[2j]_q[2-2j]_q}{[2]^2}|j,m\rangle, \tag{2.19}$$

which in the limit $q \to 1$ reduces to $j(1-j)$, the result of the action of the $SU(1,1)$ Casimir operator.

We let $EAdS^2_{q\mu}$ denote the quantum space $EAdS^2$. We also introduce the generators of the coordinate algebra $A(EAdS^2_{q\mu})$ of the space $EAdS^2_{q\mu}$ and x_+, x_-, x_3, and 1. They satisfy the commutation relations

$$\begin{aligned} &\mathbf{x}_+\mathbf{x}_- - \mathbf{x}_-\mathbf{x}_+ = -\mu\mathbf{x}_3 + (q - q^{-1})\mathbf{x}_3^2, \\ &[\mathbf{x}_3, \mathbf{x}_+]_q = q\mathbf{x}_3\mathbf{x}_+ - q^{-1}\mathbf{x}_+\mathbf{x}_3 = \mu\mathbf{x}_+, \\ &[\mathbf{x}_-, \mathbf{x}_3]_q = q\mathbf{x}_-\mathbf{x}_3 - q^{-1}\mathbf{x}_3\mathbf{x}_- = \mu\mathbf{x}_- \end{aligned} \tag{2.20}$$

and a constraint defining the space $EAdS^2_{q\mu}$

$$-\mathbf{x}_3^2 + q\mathbf{x}_-\mathbf{x}_+ + q^{-1}\mathbf{x}_+\mathbf{x}_- = -1. \tag{2.21}$$

In the space $EAdS^2_{q\mu}$, we have two different parameters q and μ, which we assume to be real. There are then three limit cases:

- the first limit $EAdS^2_{q\mu} \longrightarrow EAdS^2_\mu \longrightarrow EAdS^2$ (i.e., $q \to 1$ and then $\mu \to 0$),
- the second limit $EAdS^2_{q\mu} \longrightarrow EAdS^2_q \longrightarrow EAdS^2$ (i.e., $\mu \to 0$ and then $q \to 1$), and
- the third limit $EAdS^2_{q\mu} \longrightarrow EAdS^2$ (i.e., $\mu \to 0$ and $q \to 1$ simultaneously).

Similarly to definition (2.11), taking Casimir operator (2.19) into account, we define μ_q as

$$\mu_q \equiv \frac{[2]_q}{\sqrt{[2\ell]_q[2-2\ell]}}, \tag{2.22}$$

which becomes $1/\sqrt{\ell(1-\ell)}$ in the limit $q \to 1$.

In the third limit, constraint (2.21) becomes the well-known condition (2.2) giving the space $EAdS^2$, where $\mathbf{x}_1$ and $\mathbf{x}_2$ are the limit operators for

$$\mathbf{x}_1 = -\frac{1}{\sqrt{2}}(\mathbf{x}_+ + \mathbf{x}_-), \qquad \mathbf{x}_2 = -\frac{i}{\sqrt{2}}(\mathbf{x}_+ - \mathbf{x}_-). \tag{2.23}$$

In this case, as $q \to 1$, the operators $\mathbf{x}_i$ satisfy the relation

$$[\mathbf{x}_i, \mathbf{x}_j] = \frac{i}{\sqrt{l(1-l)}}C_{ij}{}^k\mathbf{x}_k \qquad \mathbf{x}_i\eta^{ij}\mathbf{x}_j = -1. \tag{2.24}$$

The coordinate operators $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ can be obtained from the $\mathcal{U}_q(su(1,1))$ generators $\mathbf{L}_1$, $\mathbf{L}_2$, and $\mathbf{L}_3$ as

$$\mathbf{x}_i\sqrt{[2\ell]_q[2-2\ell]_q} = \mathbf{L}_i. \tag{2.25}$$

Here, it is convenient to introduce a 2×2 matrix X formed from the generators of the quantum space $EAdS^2$:

$$X = \begin{pmatrix} -q\mathbf{x}_3 & -i(q+q^{-1})^{1/2}\mathbf{x}_+ \\ -i(q+q^{-1})^{1/2}\mathbf{x}_- & q^{-1}\mathbf{x}_3 \end{pmatrix}. \tag{2.26}$$

This matrix has the properties that it is Λ-Hermitian with $\Lambda = \sigma_3^q$, i.e., $X^* = \sigma_3^q X(\sigma_3^q)^{-1}$, and satisfies the q-deformed trace condition $\mathrm{Tr}_q X = q^{-1}X_{11} + qX_{22} = 0$.

For each irreducible unitary finite-dimensional representation of the compact semisimple Lie group $SU(2)$, we can analytically continue the representation to a finite-dimensional necessarily nonunitary representation of the noncompact semisimple Lie group $SU(1,1)$. The two Lie groups $SU(2)$ and $SU(1,1)$ have a common maximal compact subgroup $U(1)$ and a common complexification, which is the Lie group $SL(2,C)$. Considering the lowest-dimensional irreducible representation of the Lie algebra $su(2)$, which is represented by the Pauli matrices σ_1, σ_2, and σ_3, we can construct a two-dimensional nonunitary representation of the Lie algebra $su(1,1)$ as

$$\Sigma_1 = i\sigma_1, \qquad \Sigma_2 = i\sigma_2, \qquad \Sigma_3 = \sigma_3. \tag{2.27}$$

The quantum version of these matrices is

$$\begin{aligned} \Sigma_1^q &= \sqrt{\frac{[2]}{2q}}\,\Sigma_1 = \begin{pmatrix} 0 & i\sqrt{\frac{[2]}{2q}} \\ i\sqrt{\frac{[2]}{2q}} & 0 \end{pmatrix}, \\ \Sigma_2^q &= \sqrt{\frac{[2]}{2q}}\,\Sigma_2 = \begin{pmatrix} 0 & \sqrt{\frac{[2]}{2q}} \\ -\sqrt{\frac{[2]}{2q}} & 0 \end{pmatrix}, \qquad \Sigma_3^q = \begin{pmatrix} q & 0 \\ 0 & -q^{-1} \end{pmatrix}. \end{aligned} \tag{2.28}$$

We can define two new operators Σ_+^q and Σ_-^q as

$$\Sigma_+^q = \Sigma_1^q + i\Sigma_2^q = \begin{pmatrix} 0 & -i\sqrt{\frac{[2]}{q}} \\ 0 & 0 \end{pmatrix}, \qquad \Sigma_-^q = \Sigma_1^q - i\Sigma_2^q = \begin{pmatrix} 0 & 0 \\ -i\sqrt{\frac{[2]}{q}} & 0 \end{pmatrix}. \tag{2.29}$$

Using (2.26), (2.28), and (2.29), we easily obtain

$$X = \boldsymbol{\Sigma}^q \cdot \mathbf{x} = \Sigma_i^q \eta^{ij} x_j = \Sigma_1^q x_1 + \Sigma_2^q x_2 - \Sigma_3^q x_3 = \Sigma_+^q x_+ + \Sigma_-^q x_- - \Sigma_3^q x_3. \tag{2.30}$$

It is easy to see that because Σ_i^q is Σ_3-pseudo-Hermitian, i.e., $(\Sigma_i^q)^* = \Sigma_3 \Sigma_i^q \Sigma_3^{-1}$, X is a Σ_3-pseudo-Hermitian operator: $X^* = \Sigma_3 X \Sigma_3^{-1}$.

3. Spin-1/2 q-deformed pseudoprojectors of the left and right q-deformed projective $A(EAdS^2_{q\mu})$-modules

According to the Serre–Swan theorem, spaces in a noncommutative geometry are replaced with algebras of functions on these spaces. Therefore, instead of studying the principal quantum fibration $AdS^3_{q\mu} \xrightarrow{U(1)} EAdS^2_{q\mu}$, we study the noncommutative finitely generated projective module of its sections. To construct the left and right projective modules, we must construct the pseudoprojectors of these modules.

We can write the pseudoprojectors for the right projective module as

$$P^{\mathrm{R}}_{[l\pm 1/2]_q} = \frac{1}{[2]_q}\left\{1 \pm \frac{X + \mu_q/[2]_q}{\sqrt{\mu_q^2/[2]_q^2 - 1}}\right\}. \tag{3.1}$$

Here, we must require $\mu_q^2/[2]_q^2 - 1 \geq 0$, i.e., either $\mu_q \geq [2]_q$ (for the upper branch of $EAdS^2_{q\mu}$) or $\mu_q \leq -[2]_q$ (for the lower branch of $EAdS^2_{q\mu}$).

After definitions (2.22) and (2.26) are used, expression (3.1) becomes

$$P^{\mathrm{R}}_{[l\pm 1/2]_q} = \frac{\pm[2]_q \boldsymbol{\Sigma}^q \cdot \mathbf{L}^{\mathrm{R}} + [2l-1]_q \pm 1}{[2]_q[2l-1]_q}. \tag{3.2}$$

This relation connects the right angular momentum (for spin 1/2) to its maximum value $\ell + 1/2$ and its minimum value $\ell - 1/2$. It is easy to see that

$$P^{\mathrm{R}}_{[l+1/2]_q} + P^{\mathrm{R}}_{[l-1/2]_q} = \frac{2}{[2]_q}. \tag{3.3}$$

We have the projectors of our right projective $A(EAdS^2_{q\mu})$-module and therefore

$$(A(EAdS^2_{q\mu}))^2 = (A(EAdS^2_{q\mu}))^2 P^{\mathrm{R}}_{[l+1/2]_q} \oplus (A(EAdS^2_{q\mu}))^2 P^{\mathrm{R}}_{[l-1/2]_q}. \tag{3.4}$$

In the limit $q \to 1$, this reduces to the condition

$$P^{\mathrm{R}}_{l+1/2} + P^{\mathrm{R}}_{l-1/2} = 1. \tag{3.5}$$

Using (3.2), we can define the corresponding q-deformed idempotent as

$$\Gamma^{\mathrm{R}}_{[l\pm 1/2]_q} = [2]_q P^{\mathrm{R}}_{[l\pm 1/2]_q} - 1 = \frac{\pm[2]_q \boldsymbol{\Sigma}^q \cdot \mathbf{L}^{\mathrm{R}} \pm 1}{[2l-1]_q}. \tag{3.6}$$

In the limit $q \to 1$, Eqs. (3.2) and (3.6) become

$$\begin{aligned} P^{\mathrm{R}}_{l+1/2} &= \frac{\boldsymbol{\Sigma} \cdot \mathbf{L}^{\mathrm{R}} + l}{2l-1}, \qquad P^{\mathrm{R}}_{l-1/2} = \frac{-\boldsymbol{\Sigma} \cdot \mathbf{L}^{\mathrm{R}} + l - 1}{2l-1}, \\ \Gamma^{\mathrm{R}}_{l\pm 1/2} &= 2P^{\mathrm{R}}_{l\pm 1/2} - 1 = \frac{\pm 2\boldsymbol{\Sigma} \cdot \mathbf{L}^{\mathrm{R}} \pm 1}{2l-1}, \end{aligned} \tag{3.7}$$

which are the results in [21].

The projectors $P^{\mathrm{L}}_{[l\pm 1/2]_q}$ connecting the left momentum (for spin 1/2) to its maximum and minimum values $l \pm 1/2$ are obtained by changing L^{R} to $-L^{\mathrm{L}}$ in the above expressions:

$$P^{\mathrm{L}}_{[l\pm 1/2]_q} = \frac{\mp[2]_q \boldsymbol{\Sigma}^q \cdot \mathbf{L}^{\mathrm{L}} + [2l-1]_q \pm 1}{[2]_q[2l-1]_q}. \tag{3.8}$$

We also find the corresponding idempotents:

$$\Gamma^{\mathrm{L}}_{[l\pm 1/2]_q} = [2]_q P^{\mathrm{L}}_{[l\pm 1/2]_q} - 1 = \frac{\mp[2]_q \boldsymbol{\Sigma}^q \cdot \mathbf{L}^{\mathrm{L}} \pm 1}{[2l-1]_q}. \tag{3.9}$$

In the limit $q \to 1$, Eqs. (3.8) and (3.9) become

$$\begin{aligned} P^{\mathrm{L}}_{l+1/2} &= \frac{-\boldsymbol{\Sigma} \cdot \mathbf{L}^{\mathrm{L}} + l}{2l-1}, \qquad P^{\mathrm{L}}_{l-1/2} = \frac{\boldsymbol{\Sigma} \cdot \mathbf{L}^{\mathrm{L}} + l - 1}{2l-1}, \\ \Gamma^{\mathrm{L}}_{l\pm 1/2} &= 2P^{\mathrm{L}}_{l\pm 1/2} - 1 = \frac{\mp 2\boldsymbol{\Sigma} \cdot \mathbf{L}^{\mathrm{L}} \pm 1}{2l-1}. \end{aligned} \tag{3.10}$$

Here, it is again easy to see that

$$P^{\mathrm{L}}_{[l+1/2]_q} + P^{\mathrm{L}}_{[l-1/2]_q} = \frac{2}{[2]_q}. \tag{3.11}$$

We have the projectors of our left $A(EAdS^2_{q\mu})$-module and therefore

$$(A(EAdS^2_{q\mu}))^2 = (A(EAdS^2_{q\mu}))^2 P^{\mathrm{L}}_{[l+1/2]_q} \oplus (A(EAdS^2_{q\mu}))^2 P^{\mathrm{L}}_{[l-1/2]_q}. \tag{3.12}$$

Obviously,

$$\lim_{\substack{q\to 1,\\ l\to\infty}} (\Gamma^{\mathrm{L,R}}_{[l\pm 1/2]_q})^2 = 1, \tag{3.13}$$

just as we expected for the commutative chirality operator.

4. The q-deformed quantum Ginsparg–Wilson algebra and its spin-1/2 q-deformed quantum Dirac and chirality operators

The q-deformed quantum Ginsparg–Wilson algebra $\mathcal{A}_{q\mu}$ is the $*$-algebra over $\mathbb{C}$ generated by two $*$-invariant q-deformed involutions $\Gamma^{q\mu}$ and $\Gamma'^{\,q\mu}$:

$$\mathcal{A}_{q\mu} = \left\{ \langle \Gamma^{q\mu}, \Gamma'^{\,q\mu} \rangle : \lim_{q\to 1} (\Gamma^{q\mu})^2 = \lim_{q\to 1} (\Gamma'^{\,q\mu})^2 = I, \begin{array}{l} (\Gamma^{q\mu})^* = \Lambda \Gamma^{q\mu} \Lambda^{-1}, \\ (\Gamma'^{\,q\mu})^* = \Lambda \Gamma'^{\,q\mu} \Lambda^{-1} \end{array} \right\}, \tag{4.1}$$

where the operator Λ is $*$-Hermitian, involutory, and unitary:

$$\Lambda^* = \Lambda, \qquad \Lambda^2 = 1, \qquad \Lambda^{-1} = \Lambda^*. \tag{4.2}$$

Each representation of (4.1) is a realization of the q-deformed Ginsparg–Wilson algebra.

We now consider two elements constructed from the generators $\Gamma^{q\mu}_1$ and $\Gamma^{q\mu}_2$ of the q-deformed quantum Ginsparg–Wilson algebra $\mathcal{A}_{q\mu}$:

$$\begin{aligned} \Gamma^{q\mu}_1 &= \frac{1}{[2]_q}(\Gamma^{q\mu} + \Gamma'^{\,q\mu}), \qquad (\Gamma^{q\mu}_1)^* = \Lambda \Gamma^{q\mu}_1 \Lambda^{-1}, \\ \Gamma^{q\mu}_2 &= \frac{1}{[2]_q}(\Gamma^{q\mu} - \Gamma'^{\,q\mu}), \qquad (\Gamma^{q\mu}_1)^* = \Lambda \Gamma^{q\mu}_2 \Lambda^{-1}. \end{aligned} \tag{4.3}$$

The elements $\Gamma^{q\mu}_1$ and $\Gamma^{q\mu}_2$ anticommute, $\{\Gamma^{q\mu}_1, \Gamma^{q\mu}_2\} = 0$. Respectively identifying $\Gamma^{\mathrm{L}}_{[l+1/2]_q}$ and $\Gamma^{\mathrm{R}}_{[l+1/2]_q}$ with $\Gamma^{q\mu}_1$ and $\Gamma^{q\mu}_2$, we obtain

$$\Gamma^{q\mu}_1 = \frac{1}{[2]_q} \frac{[2]_q \boldsymbol{\Sigma}^q \cdot \boldsymbol{\mathcal{L}}^q - 2}{[2l-1]_q}, \qquad \Gamma^{q\mu}_2 = \frac{1}{[2]_q} \frac{[2]_q \boldsymbol{\Sigma}^q \cdot (\mathbf{L}^{\mathrm{L}} + \mathbf{L}^{\mathrm{R}})}{[2l-1]_q}, \tag{4.4}$$

where $\boldsymbol{\mathcal{L}}^q$ is the q-deformed quantum version of the angular momentum operator.

We now define the Σ_3-pseudo-Hermitian q-deformed quantum Dirac and chirality operators on the q-deformed quantum space $EAdS^2$ as

$$D_{q\mu} = [2l-1]_q \Gamma^{q\mu}_1 = \frac{1}{[2]_q}([2]_q \boldsymbol{\Sigma}^q \cdot \boldsymbol{\mathcal{L}}^q - 2), \qquad \gamma_{q\mu} = \Gamma^{q\mu}_2. \tag{4.5}$$

We take the abovementioned third limit:

$$\lim_{q\to 1,\, \mu\to 0} D_{q\mu} = \boldsymbol{\sigma} \cdot \boldsymbol{\mathcal{L}} - 1, \qquad \lim_{q\to 1,\, \mu\to 0} \gamma_{q\mu} = \boldsymbol{\sigma} \cdot \mathbf{x}. \tag{4.6}$$

These are the correct Σ_3-pseudo-Hermitian Dirac and chirality operators on the commutative space $EAdS^2$. It is also easy to see that

$$\lim_{\substack{q\to 1,\\ \mu\to 0}} \{D_{q\mu}, \gamma_{q\mu}\} = 0, \tag{4.7}$$

just as we expected for the Dirac and chirality operators on $EAdS^2$.

Our operators $D_{q\mu}$ and $\gamma_{q\mu}$ are not the only Dirac and chirality operators on $EAdS^2_{q\mu}$. In a noncommutative geometry, the right and left $A(EAdS^2_{q\mu})$-modules are not isomorphic. Because the left and right momentum operators are not equivalent, we can choose another set of operators to construct our q-deformed Ginsparg–Wilson algebra. Choosing $\Gamma^{\mathrm{L}}_{[l+1/2]_q}$ and $\Gamma^{\mathrm{R}}_{[l-1/2]_q}$ and considering

$$[2l-1]_q(\Gamma^{\mathrm{L}}_{[l+1/2]_q} - \Gamma^{\mathrm{R}}_{[l-1/2]_q}),$$

we obtain the correct Dirac operator (4.6) in the limit case. The corresponding chirality operator is obtained from

$$\frac{1}{[2]_q}(\Gamma^{\mathrm{L}}_{[l+1/2]_q} + \Gamma^{\mathrm{R}}_{[l-1/2]_q})$$

because we also have a correct limit (4.6) in this case.

Another possibility is to combine $\Gamma^{\mathrm{L}}_{[l-1/2]_q}$ and $\Gamma^{\mathrm{R}}_{[l+1/2]_q}$. We define

$$-[2l-1]_q(\Gamma^{\mathrm{L}}_{[l-1/2]_q} - \Gamma^{\mathrm{R}}_{[l+1/2]_q}), \qquad \frac{1}{[2]_q}(\Gamma^{\mathrm{L}}_{[l-1/2]_q} + \Gamma^{\mathrm{R}}_{[l+1/2]_q})$$

as the respective Dirac and chirality operators. This exhausts all possible combinations.

5. Conclusion

Using q-deformed quantum pseudoprojectors and idempotents of a finitely generated q-deformed projective $A(EAdS^2_{q\mu})$-module, we have constructed the generators of the q-deformed quantum Ginsparg–Wilson algebra. Using this algebra, we constructed q-deformed quantum Dirac and chirality operators. The value of these operators is that they have the correct commutative limit.

Conflicts of interest. The authors declare no conflicts of interest.

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