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Gauged Dirac operator on the quantum sphere in instanton sector

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In this paper, we construct the q-deformed fuzzy Dirac and chirality operators on quantum fuzzy Podles sphere $S^2_{q\mu}$. Using the q-deformed fuzzy Ginsparg–Wilson algebra, we study the q-deformed gauged fuzzy Dirac and chirality operators in instanton sector. We will show the correct fuzzy sphere limit in the limit case $q \to 1$ and the correct commutative limit in the limit case when $q \to 1$ and noncommutative parameter l tends to infinity.

Keywords: Quantum fuzzy sphere; gauged q-deformed Ginsparg–Wilson algebra; gauged q-deformed Dirac and chirality operators.

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1. Introduction

Dirac and chirality operators are two important self-adjoint operators for the Connes–Lott approach to noncommutative geometry. A unital spectral triple,[1,2] $(\mathcal{A}, \mathcal{H}, D)$ consists of a complex unital $*$-algebra $\mathcal{A}$, faithfully $*$-represented by bounded operators on a separable Hilbert space $\mathcal{H}$, and a self-adjoint operator $D : \mathcal{H} \to \mathcal{H}$ (the Dirac operator) with the following properties:

- the resolvent $(D - \lambda)^{-1}$, $\lambda \notin \mathbb{R}$ is a compact operator on $\mathcal{H}$,
- for all $a \in \mathcal{A}$ the commutator $[D, \pi(a)]$ is a bounded operator on $\mathcal{H}$.

A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is called even if there exists a $\mathbb{Z}_2$-grading of $\mathcal{H}$, i.e. an operator $\gamma : \mathcal{H} \to \mathcal{H}$ with $\gamma^* = \gamma$ and $\gamma^2 = 1$, such that $\gamma D + D\gamma = 0$ and $\gamma a = a\gamma$ for all $a \in \mathcal{A}$. Otherwise the spectral triple is said to be odd. For odd-dimensional manifolds, there are no chirality operators and in such a case, the Dirac operator describes only the differential structures. There are three types of Dirac and chirality operators on the fuzzy two-sphere. Ginsparg–Wilson Dirac operator,

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$D_{\rm GW}$,[3–10] Watamura–Watamura Dirac operator $D_{\rm WW}$[11–13] and Grosse–Klimcik–Presnajder Dirac operator $D_{\rm GKP}$.[14,15] These three types of Dirac operators are compared with each other in Ref. 16. The idea of q-deformed geometry was extensively studied in the late 1980s and 1990s. The q-deformed Hopf fibration has been studied in the framework of Hopf–Galois extension in Ref. 17. Podles sphere are introduced in Refs. 18 and 19. The q-deformed Dirac operator on quantum Podles sphere has been studied from different approaches.[20–26] The q-deformed Watamura–Watamura Dirac operator $D^q_{\rm WW}$, has been studied in Ref. 20. The authors constructed Dirac and chirality operators on noncommutative space having $U_q(su(2))$ as the symmetry group. It has been argued that the Dirac operator is covariant and in the commutative limit where the underlying space is Podles sphere, the full rotational invariance of the Dirac operator is recovered. It was further shown that the Dirac operator reduces to that obtained in Ref. 12. In Ref. 22 the goal of the authors is to describe q-deformed version of GKP Dirac operator on quantum sphere. In Ref. 5 it has been showed that how one can construct gauged Dirac operator in instanton sector on fuzzy sphere. In this paper our aim is to study the gauged q-deformed fuzzy Ginsparg–Wilson Dirac and chirality operators in instanton sector using the gauged q-deformed fuzzy Ginsparg–Wilson algebra on quantum fuzzy two-sphere. This paper is organized as follows. In Sec. 2 we briefly review the first Hopf fibration $S^3 \to S^2$ with special focus on projectors and projective modules of the sections of this principal bundle. Also, we study the q-deformed fuzzy version of the first Hopf fibration. Specially we study the q-deformed projectors and q-deformed projective modules of the Hopf–Galois extension of the fibration $S^3 \to S^2$. In Sec. 3 spin $\frac{1}{2}$ q-deformed projectors of the left and right q-deformed projective $A\big(S^2_{q\mu}\big)$-module will be studied. The q-deformed fuzzy Ginsparg–Wilson algebra is constructed in Sec. 4. Also, in this section we construct q-deformed fuzzy chirality and Dirac operators using the q-deformed left and right projectors and their corresponding idempotents. In Sec. 5 gauged q-deformed fuzzy Dirac operator is studied. In Sec. 6 the q-deformed fuzzy Dirac operator in instanton sector is constructed. Finally, in Sec. 7, we look into gauging the q-deformed fuzzy Dirac operator in instanton sector. In each step, we compare our results with the limit case $q \to 1$.

2. Theoretical Formalism

Consider the $U(1)$ principal fibration π with $S^3 \cong SU(2)$ as total space over the base space S^2:

$$U(1) \xrightarrow{\text{right } U(1)\text{-action}} S^3 \xrightarrow{\pi} S^2 \,. \tag{2.1}$$

Let $B_{\mathbb{C}} = C^\infty(S^3, \mathbb{C})$ and $A_{\mathbb{C}} = C^\infty(S^2, \mathbb{C})$ denote the algebras of $\mathbb{C}$-valued smooth functions on the total manifold S^3 and base manifold S^2 under point-wise multiplication, respectively. The elements of $B_{\mathbb{C}}$ can be classified into the right modules,

$$C^\infty_n(S^3, \mathbb{C}) = \{\varphi : S^3 \to \mathbb{C},\ \varphi(p \cdot \omega) = \omega^{-n} \cdot \varphi(p),\ \forall p \in S^3,\ \forall \omega \in U(1)\}\,, \tag{2.2}$$

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over the pullback of the $A_{\mathbb{C}}$. The left actions of the group $U(1)$ on complex numbers are labeled by an integer n which characterizes the bundle. The Serre–Swan theorem[27] states that for a compact smooth manifold, there is an equivalence between the smooth vector bundles over that manifold and the finitely generated projective modules. In algebraic K-theory, it is well known that corresponds to these bundles, there are pseudo-projectors P_n (Ref. 28) such that, for the associated vector bundle

$$E^{(n)} = S^3 \times_{U(1)} \mathbb{C} \xrightarrow{\pi} S^2\,, \tag{2.3}$$

right $A_{\mathbb{C}}$-module of sections $\Gamma^\infty\big(S^2, E^{(n)}\big)$ which is isomorphic with $C^\infty_{(n)}\big(S^3, \mathbb{C}\big)$ is equivalent to the image in the free module $(A_{\mathbb{C}})^{(n+1)} = C^\infty(S^2,\mathbb{C}) \otimes \mathbb{C}^{n+1}$ of a projector P_n, $\Gamma^\infty\big(S^2, E^{(n)}\big) = P_n(A_{\mathbb{C}})^{n+1}$. The projector P_n is a Hermitian operator of rank 1 over $\mathbb{C}$:

$$P_n \in M_{n+1}(A_{\mathbb{C}})\,, \quad P_n^2 = P_n\,, \quad P_n^\dagger = P_n\,, \quad \mathrm{Tr}\, P_n = 1\,. \tag{2.4}$$

For the right $A_{\mathbb{C}}$-module of sections $\Gamma^\infty\big(S^2, E^{(n)}\big)$ there exist $n+1$ projectors $P_1, P_2, \ldots, P_{n+1}$ having the same rank 1. Therefore the free module $(A_{\mathbb{C}})^{n+1}$ can be written as a direct sum of the projective $A_{\mathbb{C}}$-modules,

$$(A_{\mathbb{C}})^{n+1} = \bigoplus_{i=1}^{n+1} P_i(A_{\mathbb{C}})^{n+1}\,. \tag{2.5}$$

Noncommutative geometry is a pointless geometry. In this geometry instead of the coordinates x_i's of S^2, The angular momentum generators in the unitary irreducible ℓ-representation space have the role of the points of the fuzzy S^2. Let us denote the angular momentum operator on the commutative S^2 by $\mathbf{L}$ and the fuzzy Hermitian matrix algebra by $\mathcal{A}_l = \{\alpha \in M_{2l+1}(\mathbb{C})\}$. Every arbitrary element α can be expressed in terms of the bases, as a unitary l-representation of the $su(2)$, of the angular momentum $\mathbf{L} = (L_1, L_2, L_3)$:

$$[L_i, L_j] = i\epsilon_{ijk} L_k\,. \tag{2.6}$$

In the Hopf fibration $S^3 \xrightarrow{U(1)} S^2$, the module of sections is $C(S^2)$-module $\Gamma^\infty\big(S^2, E^{(n)}\big)$ in which $C(S^2)$ is the commutative algebra of functions on S^2. In the fuzzy case, this algebra is a noncommutative algebra and therefore left and right modules are not isomorphic. In this case to each angular momentum operator $\mathbf{L}$, We associate two linear operators $\mathbf{L}^L$ and $\mathbf{L}^R$ with the left and right actions on $\mathcal{A}_l$:

$$L_i^L \alpha = L_i \alpha\,, \quad L_i^R \alpha = \alpha L_i\,, \quad \forall \alpha \in \mathcal{A}_l\,, \tag{2.7}$$

where these left and right operators commute with each other:

$$\big[L_i^L, L_j^R\big] = 0\,. \tag{2.8}$$

The L_i^L and L_j^R have the same $su(2)$ algebra with the Casimir C:

$$C \equiv \mathbf{L}^L \cdot \mathbf{L}^L = \mathbf{L}^R \cdot \mathbf{L}^R = l(l+1)1\,. \tag{2.9}$$

We use $\mathbf{L}^L$, $\mathbf{L}^R$ to define the fuzzy version of orbital momentum operator $\boldsymbol{\mathcal{L}}$ on the fuzzy S^2. We define $\boldsymbol{\mathcal{L}}$ by the adjoint action of L_i on the $\mathcal{A}_l$:

$$\mathcal{L}_i\alpha = \big(L_i^L - L_i^R\big)\alpha = \mathrm{ad}_{L_i}\,\alpha = [L_i, \alpha]\,. \tag{2.10}$$

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The hypersurface S^2 can be described by the coordinates x_i

$$\mathbf{x}\cdot\mathbf{x} = x_1^2 + x_2^2 + x_3^2 = 1\,. \tag{2.11}$$

The commutative coordinates x_i as a limiting case are obtained from the fuzzy S^2. Using the Casimir of the $su(2)$ algebra we define the parameter μ as

$$\mu = \frac{1}{\sqrt{l(l+1)}} \tag{2.12}$$

Equation (2.6) becomes

$$[x_i, x_j] = \frac{i}{\sqrt{l(l+1)}}\epsilon_{ijk}x_k = i\mu\epsilon_{ijk}x_k\,. \tag{2.13}$$

The commutative coordinates x_i as a limiting case are obtained from the fuzzy S^2:

$$x_i = \lim_{l\to\infty}\frac{L_i^{L,R}}{\sqrt{l(l+1)}} = \lim_{l\to\infty}\frac{L_i^{L,R}}{l}\,. \tag{2.14}$$

The quantum Hopf bundle is a $U(1)$-bundle over the standard Podles sphere S_q^2 and whose total manifold is the manifold of the quantum group $SU_q(2)$. The sphere S_q^2 is a quantum homogeneous space $SU_q(2)$. Let us denote by $A(SU_q(2))$, $A(S_q^2)$ and $A(U(1))$ the coordinate algebra of the total space $SU_q(2)$, base space S_q^2 and the fibre $U(1)$, respectively.

The coordinate algebra $A(SU_q(2))$ and the quantum universal enveloping algebra $\mathcal{U}_q(su(2)) \equiv su_q(2)$ are $*$-algebra and Hopf $*$-algebra, respectively.[17]

The algebra inclusion $A(S_q^2) \hookrightarrow A(SU_q(2))$ which is a Hopf–Galois extension is a quantum Hopf bundle. It is the algebraic version of the first quantum Hopf fibration $SU_q(2) \xrightarrow{U_q(1)} S_q^2$. The quantum associated bundle is given by[29]

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$$E_q^{(n)} := \Big\{x \in A(SU_q(2)) :\ k\cdot x = q^{\frac{n}{2}}x,\ k \in A(U_q(1))\Big\}\,. \tag{2.15}$$

Each $E_q^{(n)}$ is clearly a $A(S_q^2)$-bi-module and is equivalent to the image in the free module $\big(A(S_q^2)\big)^{n+1}$ of a self-adjoint quantum projector $p_q^{(n)}$ in $\mathrm{Mat}_{n+1}\big(A(S_q^2)\big)$ (for $n \geq 0$). We identify $E_q^{(n)}$ with the left $A(S_q^2)$-module of sections $\big(A(S_q^2)\big)^{n+1}p_q^{(n)}$.

The algebra $U_q(su(2)) \equiv su_q(2)$ is generated by elements $J_\pm$ and J_3 satisfying the relation

$$[J_+, J_-] = [J_3]_q\,, \quad J_3J_\pm J_3^{-1} = q^{\pm}J_{\pm 1}\,, \quad J_3^* = J_3\,, \quad J_\pm^* = J_\mp\,, \tag{2.16}$$

where the $*$-structure is for real q. The structure of the Hopf algebra is given by

$$\triangle(J_\pm) = J_\pm \otimes J_3^{-1} + J_3 \otimes J_\pm\,, \quad \triangle(J_3) = J_3 \otimes J_3\,. \tag{2.17}$$

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The generators of $U_q(su(2))$ act on the spin j representation in the following way

$$J_\pm|j,m\rangle = ([j \mp m]_q [j \pm m + 1]_q)^{\frac{1}{2}} |j, m \pm 1\rangle \,, \quad J_3|j,m\rangle = q^m |j,m\rangle \,, \tag{2.18}$$

where $-j \leq m \leq j$ and

$$[n]_q \equiv \frac{q^n - q^{-n}}{q - q^{-1}} \,. \tag{2.19}$$

The value of the quantum Casimir operator C_q is given by[26]

$$C_q|j,m\rangle = \frac{[2j]_q [2j+2]_q}{[2]^2} |j,m\rangle \,, \tag{2.20}$$

where in the $q \to 1$ limit, it reduces to $j(j+1)$, the value of the $SU(2)$ Casimir operator. Let us denote by $S^2_{q\mu}$ the fuzzy quantum two-sphere. Also, we denote the generators of the coordinate algebra of $S^2_{q\mu}$, i.e. $A(S^2_{q\mu})$ by x_+, x_-, x_3, along with the unit 1. They satisfy the commutation relations:[20,21]

$$\mathbf{x}_+\mathbf{x}_- - \mathbf{x}_-\mathbf{x}_+ = \mu\mathbf{x}_3 - (q - q^{-1})x_3^2 \,, \quad [\mathbf{x}_3, \mathbf{x}_\pm]_q = \pm\mu\mathbf{x}_\pm \,, \tag{2.21}$$

with the sphere constraint

$$x_3^2 + q x_- x_+ + q^{-1} x_+ x_- = 1 \,. \tag{2.22}$$

In Eq. (2.21), we define the quantum bracket as

$$[A, B]_q = qAB - q^{-1}BA \,. \tag{2.23}$$

In the quantum fuzzy sphere $S^2_{q\mu}$ we have two different parameter q and μ, which we take to be real. Then, there are different limiting cases. These different limits can be expressed as:

- *First limit*: $S^2_{q\mu} \to S^2_\mu \to S^2$, ($q \to 1$ followed by $\mu \to 0$),
- *Second limit*: $S^2_{q\mu} \to S^2_q \to S^2$, ($\mu \to 0$ followed by $q \to 1$),
- *Third limit*: $S^2_{q\mu} \to S^2$, ($\mu \to 0$, $q \to 1$ simultaneously),

where $S^2_{q\mu}$ is the fuzzy sphere. Similar to definition (2.12), using the Casimir (2.20), we define μ_q as

$$\mu_q \equiv \frac{[2]_q}{\sqrt{[2l]_q[2l+2]}} \,, \tag{2.24}$$

where in the limit $q \to 1$ it becomes $\frac{1}{\sqrt{l(l+1)}}$.

In the third limit case the constraint (2.22) becomes the well-known two-sphere S^2:

$$x_1^2 + x_2^2 + x_3^2 = 1 \,. \tag{2.25}$$

where $\mathbf{x}_1$ and $\mathbf{x}_2$ are limit case of

$$\mathbf{x}_1 = -\frac{1}{\sqrt{2}}(\mathbf{x}_+ + \mathbf{x}_-) \,, \quad \mathbf{x}_2 = -\frac{1}{\sqrt{2}}(\mathbf{x}_+ - \mathbf{x}_-) \,. \tag{2.26}$$

In the case $q \to 1$ $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ satisfies:

$$[\mathbf{x}_i, \mathbf{x}_j] = -\frac{i}{\sqrt{l(l+1)}}\epsilon_{ijk}\mathbf{x}_k\,. \tag{2.27}$$

The coordinate operators $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ can be obtained from the $\mathcal{U}_q(su(2))$ generators $\mathbf{L}_1$, $\mathbf{L}_2$, $\mathbf{L}_3$ as

$$\mathbf{x}_i\sqrt{[2l]_q[2l+2]_q} = \mathbf{L}_i\,. \tag{2.28}$$

Also, the commutative coordinates x_i of commutative S^2 can be given by the following limits:

$$x_i = \lim_{q\to 1,\, \mu\to 0} \frac{L_i}{\sqrt{[2\ell]_q[2l+2]_q}}\,. \tag{2.29}$$

Now, let us define the 2×2 matrix X as

$$X = \begin{pmatrix} q\mathbf{x}_3 & -\sqrt{\frac{[2]_q}{q}}\mathbf{x}_+ \\ -\sqrt{\frac{[2]_q}{q}}\mathbf{x}_- & -q^{-1}\mathbf{x}_3 \end{pmatrix} \tag{2.30}$$

Let, $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2$, $\boldsymbol{\sigma}_3$, be Pauli matrices of $su(2)$ Lie algebra. Then, we can define the q-deformed version of these matrices as

$$\begin{aligned}\Sigma_1^q &= \sqrt{\frac{[2]_q}{2q}}\sigma_1 = \sqrt{\frac{[2]_q}{2q}}\begin{pmatrix} 0 & i \\ i & 0\end{pmatrix},\\ \Sigma_2^q &= \sqrt{\frac{[2]_q}{2q}}\sigma_2 = \sqrt{\frac{[2]_q}{2q}}\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix},\\ \Sigma_3^q &= \begin{pmatrix} q & 0 \\ 0 & -q^{-1}\end{pmatrix}.\end{aligned} \tag{2.31}$$

Now, using (2.30) and (2.31) it is easy to see that

$$X = {\Sigma_i}^q x_i = \boldsymbol{\Sigma}^q\cdot\mathbf{x} = \Sigma_+^q\mathbf{x}_+ + \Sigma_-^q\mathbf{x}_- + \Sigma_3^q\mathbf{x}_3\,, \tag{2.32}$$

where

$$\Sigma_+^q = \Sigma_1^q + i\Sigma_2^q\,,\quad \Sigma_-^q = \Sigma_1^q - i\Sigma_2^q\,. \tag{2.33}$$

3. Spin $\frac{1}{2}$ q-Deformed Projectors of the Left and Right q-Deformed Projective $A(S^2_{q\mu})$-Module

According to the Serre–Swan's theorem, in noncommutative geometry, manifolds are replaced with the algebra of functions on them.[27] Therefore, study of the quantum principal fibration $SU_q(2) \xrightarrow{U_q(1)} S_q^2$, replaces with the study of noncommutative finitely generated projective $A(S^2_{q\mu})$-module of its sections. To build the left and right q-deformed projective modules we should construct the fuzzy q-projectors of these modules.

The projectors for right projective module can be written as

$$P^R_{[l\pm\frac{1}{2}]_q} = \frac{1}{[2]_q}\left\{1 \mp \frac{\left(X - \frac{\mu_q}{[2]_q}\right)}{\sqrt{\frac{\mu^2}{[2]_q^2}+1}}\right\}. \tag{3.1}$$

Substituting (2.24) and (2.30) in (3.1) we can write

$$P^R_{[l-\frac{1}{2}]_q} = \frac{[2]_q\mathbf{\Sigma}^q\cdot\mathbf{L}^R + [2l+1]_q - 1}{[2]_q[2l+1]_q}\,, \tag{3.2}$$

which couples left angular momentum and spin $\frac{1}{2}$ to its minimum value $l-\frac{1}{2}$, and

$$P^R_{[l+\frac{1}{2}]_q} = \frac{-[2]_q\mathbf{\Sigma}^q\cdot\mathbf{L}^R + [2l+1]_q + 1}{[2]_q[2l+1]_q}\,, \tag{3.3}$$

which couples left angular momentum and spin $\frac{1}{2}$ to its maximum value $l+\frac{1}{2}$. Here, $\mathbf{\Sigma}^q_i$ are q-deformed Pauli matrices and $\mathbf{L}^R_i$ are right q-deformed angular momentum.

It is easy to see that

$$P^R_{[l+\frac{1}{2}]_q} + P^R_{[l-\frac{1}{2}]_q} = \frac{2}{[2]_q}\,. \tag{3.4}$$

These are the projectors of our right projective $A(S^2_{q\mu})$-module

$$\left(A(S^2_{q\mu})\right)^2 = \left(A(S^2_{q\mu})\right)^2 P^R_{[l+\frac{1}{2}]_q} \oplus \left(A(S^2_{q\mu})\right)^2 P^R_{[l-\frac{1}{2}]_q}\,,$$

which in the limit case $q\to 1$ reduces to the condition:

$$P^R_{l+\frac{1}{2}} + P^R_{l-\frac{1}{2}} = 1\,. \tag{3.5}$$

Using (3.2) and (3.3) we can define the corresponding q-deformed idempotent as

$$\Gamma^R_{[l-\frac{1}{2}]_q} = [2]_q P^R_{[l-\frac{1}{2}]_q} - 1 = \frac{[2]_q\mathbf{\Sigma}^q\cdot\mathbf{L}^R - 1}{[2l+1]_q}\,, \tag{3.6}$$

$$\Gamma^R_{[l+\frac{1}{2}]_q} = [2]_q P^R_{[l+\frac{1}{2}]} - 1 = \frac{-[2]_q\mathbf{\Sigma}^q\cdot\mathbf{L}^R + 1}{[2l+1]_q}\,. \tag{3.7}$$

In the limit $q\to 1$, Eqs. (3.2), (3.3), (3.6) and (3.7) become

$$\begin{aligned} P^R_{l+\frac{1}{2}} &= \frac{\sigma\cdot\mathbf{L}^R + l + 1}{2l+1}\,, \quad P^R_{l-\frac{1}{2}} = \frac{\sigma\cdot\mathbf{L}^R + l}{2l+1}\,,\\ \Gamma^R_{l+\frac{1}{2}} &= \frac{-\sigma\cdot\mathbf{L}^R + \frac{1}{2}}{l+\frac{1}{2}}\,, \quad \Gamma^R_{l-\frac{1}{2}} = \frac{\sigma\cdot\mathbf{L}^R - \frac{1}{2}}{l+\frac{1}{2}}\,. \end{aligned} \tag{3.8}$$

which are the results of Ref. 4. The projector $P^L_{[l-\frac{1}{2}]_q}$ coupling the left momentum and spin $\frac{1}{2}$ to its minimum value $l-\frac{1}{2}$ is obtained by changing L^R to $-L^L$ in the above expression

$$P^L_{[l-\frac{1}{2}]_q} = \frac{-[2]_q \mathbf{\Sigma}^q \cdot \mathbf{L}^L + [2l+1]_q - 1}{[2]_q [2l+1]_q} \tag{3.9}$$

and $P^L_{[l+\frac{1}{2}]_q}$ is given by

$$P^L_{[l+\frac{1}{2}]_q} = \frac{[2]_q \mathbf{\Sigma}^q \cdot \mathbf{L}^L + [2l+1]_q + 1}{[2]_q [2l+1]_q}, \tag{3.10}$$

which couples left momentum and spin $\frac{1}{2}$ to its maximum value $l + \frac{1}{2}$.

Here, again it is easy to see that

$$P^L_{[l+\frac{1}{2}]_q} + P^L_{[l-\frac{1}{2}]_q} = \frac{2}{[2]_q}. \tag{3.11}$$

These are the the projectors are our left $A(S^2_{q\mu})$-module

$$\big(A(S^2_{q\mu})\big)^2 = \big(A(S^2_{q\mu})\big)^2 P^L_{[l+\frac{1}{2}]_q} \oplus \big(A(S^2_{q\mu})\big)^2 P^L_{[l-\frac{1}{2}]_q}.$$

The corresponding quantum idempotents are

$$\Gamma^L_{[l-\frac{1}{2}]_q} = [2]_q P^L_{[l-\frac{1}{2}]_q} - 1 = -\frac{[2]_q \mathbf{\Sigma}^q \cdot \mathbf{L}^L + 1}{[2l+1]_q}, \tag{3.12}$$

$$\Gamma^L_{[l+\frac{1}{2}]_q} = [2]_q P^L_{[l+\frac{1}{2}]} - 1 = \frac{[2]_q \mathbf{\Sigma}^q \cdot \mathbf{L}^L + 1}{[2l+1]_q}. \tag{3.13}$$

It is clear that

$$\lim_{q\to 1,\, l\to\infty} \Big(\Gamma^{L,R}_{[l\pm\frac{1}{2}]}\Big)^2 = 1. \tag{3.14}$$

In the limit $q \to 1$, Eqs. (3.9), (3.10), (3.12) and (3.13) become

$$P^L_{l+\frac{1}{2}} = \frac{\sigma \cdot \mathbf{L}^L + l + 1}{2l+1}, \qquad P^L_{l-\frac{1}{2}} = \frac{-\sigma \cdot \mathbf{L}^L + l}{2l+1}, \tag{3.15}$$

$$\Gamma^L_{l+\frac{1}{2}} = 2P^L_{l+\frac{1}{2}} - 1 = \frac{\sigma \cdot \mathbf{L}^L + \frac{1}{2}}{l + \frac{1}{2}}, \quad \Gamma^L_{l-\frac{1}{2}} = 2P^L_{l-\frac{1}{2}} - 1 = -\frac{\sigma \cdot \mathbf{L}^L + \frac{1}{2}}{l + \frac{1}{2}}. \tag{3.16}$$

which are the results of fuzzy sphere limit of Ref. 4.

4. *q*-Deformed Fuzzy Ginsparg–Wilson Algebra and its Spin $\frac{1}{2}$ *q*-Deformed Fuzzy Dirac and Chirality Operators

The q-deformed fuzzy Ginsparg–Wilson algebra $\mathcal{A}_{q\mu}$ is the $*$-algebra (here $* = \dagger$) over $\mathbb{C}$, generated by two $*$-invariant q-deformed involution $\Gamma^{q\mu}$ and $\Gamma'^{q\mu}$:

$$\mathcal{A}_{q\mu} = \langle \Gamma^{q\mu}, \Gamma'^{q\mu} \colon\ (\Gamma^{q\mu})^2 = (\Gamma'^{q\mu})^2 = I,\ (\Gamma^{q\mu})^\dagger = \Gamma^{q\mu},\ (\Gamma'^{q\mu})^\dagger = \Gamma'^{q\mu} \rangle, \tag{4.1}$$

each representation of (4.1) is a realization of the q-deformed Ginsparg–Wilson algebra. Now, consider the following two elements constructed out of the generators $\Gamma^{q\mu}$ and $\Gamma'^{q\mu}$ of the q-deformed fuzzy Ginsparg–Wilson algebra $\mathcal{A}_{q\mu}$:

$$\begin{aligned}\Gamma_1^{q\mu} &= \frac{1}{[2]_q}\left(\Gamma^{q\mu} + \Gamma'^{q\mu}\right), \quad \left(\Gamma_1^{q\mu}\right)^* = \Gamma_1^{q\mu},\\ \Gamma_2^{q\mu} &= \frac{1}{[2]_q}\left(\Gamma^{q\mu} - \Gamma'^{q\mu}\right), \quad \left(\Gamma_2^{q\mu}\right)^* = \Gamma_2^{q\mu}.\end{aligned} \tag{4.2}$$

So that, $\Gamma_1^{q\mu}$ and $\Gamma_2^{q\mu}$ anticommute with each other:

$$\left\{\Gamma_1^{q\mu}, \Gamma_2^{q\mu}\right\} = 0. \tag{4.3}$$

Identifying $\Gamma^L_{[l+\frac{1}{2}]}$ and $\Gamma^R_{[l+\frac{1}{2}]}$ with $\Gamma^{q\mu}$ and $\Gamma'^{q\mu}$, we get

$$\Gamma_1^{q\mu} = \frac{1}{[2]_q}\left(\frac{[2]_q \boldsymbol{\Sigma}^q \cdot \mathcal{L}^q + 2}{[2l+1]_q}\right), \quad \Gamma_2^{q\mu} = \frac{1}{[2]_q}\left(\frac{[2]_q \boldsymbol{\Sigma}^q \cdot (\mathbf{L}^L + \mathbf{L}^R)}{[2l+1]_q}\right), \tag{4.4}$$

where $\mathcal{L}^q$ is the q-deformed fuzzy version of angular momentum operator.

Now, let us define the q-deformed fuzzy Dirac and chirality operators on q-deformed fuzzy sphere as

$$D_{q\mu} = [2l+1]_q \Gamma_1^{q\mu} = \frac{1}{[2]_q}\left([2]_q \boldsymbol{\Sigma}^q \cdot \mathcal{L}^q + 2\right), \quad \gamma_{q\mu} = \Gamma_2^{q\mu}. \tag{4.5}$$

Here, let us apply the third limit case mentioned before to (4.5):

$$\lim_{q\to 1, \mu \to 0} D_{q\mu} = \sigma \cdot \mathcal{L} + 1, \quad \lim_{q\to 1, \mu\to 0} \gamma_{q\mu} = \sigma \cdot \mathbf{x}. \tag{4.6}$$

These are the Dirac and chirality operators on commutative S^2. Also, it is easy to see that

$$\lim_{q\to 1, \mu\to 0} \{D_{q\mu}, \gamma_{q\mu}\} = 0, \tag{4.7}$$

which we expect from Dirac and chirality operators on S^2.

These $D_{q\mu}$ and $\gamma_{q\mu}$ are not the only Dirac and chirality operators on $S^2_{q\mu}$. In noncommutative geometry, the right and left $\mathcal{A}(S^2_{q\mu})$-modules are not isomorphic. Because the left and right momentum operators are not equivalent, we can choose another set of operators to construct our q-deformed Ginsparg–Wilson algebra.

Choosing $\Gamma^L_{[l+\frac{1}{2}]_q}$ and $\Gamma^R_{[l-\frac{1}{2}]_q}$ and considering

$$[2l+1]_q\left(\Gamma^L_{[l+\frac{1}{2}]_q} - \Gamma^R_{[l-\frac{1}{2}]_q}\right),$$

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we get the correct Dirac operator (4.6) in the limit case. The corresponding chirality operator is got from

$$\frac{1}{[2]_q}\left(\Gamma^L_{[l+\frac{1}{2}]_q} + \Gamma^R_{[l-\frac{1}{2}]_q}\right)$$

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as this goes to the correct limit case (4.6). The other possibility is to combine $\Gamma^L_{[l-\frac{1}{2}]_q}$ and $\Gamma^R_{[l+\frac{1}{2}]_q}$, we define

$$-[2l+1]_q\left(\Gamma^L_{[l-\frac{1}{2}]_q}-\Gamma^R_{[l+\frac{1}{2}]_q}\right) \quad \text{and} \quad \frac{1}{[2]_q}\left(\Gamma^L_{[l-\frac{1}{2}]_q}+\Gamma^R_{[l+\frac{1}{2}]_q}\right)$$

as Dirac and chirality operators, respectively. These are all the possible combinations.

5. *q*-Deformed Fuzzy Gauged Dirac Operator (No Instanton Fields)

Let us start with $S^2_{q\mu}(2l+1)\otimes\mathbb{C}^k$. The $*$-invariant fuzzy gauge field A^L_i acts on $\xi=(\xi_1,\ldots,\xi_k)$, $\xi_i\in S^2_{q\mu}\,(2\ell+1)$ as

$$\left(A^L_i\xi\right)_m=(A_i)_{mn}\xi_n\,. \tag{5.1}$$

The $*$-invariant condition on A^L_i is

$$\left(A^L_i\right)^*=A^L_i\,, \tag{5.2}$$

which on the commutative S^2 becomes a commutative field $\mathbf{a}$ and its components a_i have to be tangent to commutative S^2:

$$\mathbf{x}\cdot\mathbf{a}=0\,. \tag{5.3}$$

We need a condition to get the above result for large l. One of the conditions of such a nature is[26]

$$\begin{aligned}\left(\mathbf{L}^L+\mathbf{A}^L\right)\cdot\left(\mathbf{L}^L+\mathbf{A}^L\right)&=\sum q^m\left(L^L+A^L\right)_{-m}\left(L^L+A^L\right)_m\\&=\mathbf{L}^L\cdot\mathbf{L}^L=\frac{[2l]_q[2l+2]_q}{[2]^2}\,.\end{aligned} \tag{5.4}$$

where m runs over $0,\pm1$. In the following relations we use the Einstein's summation convention. The expansion of (5.4) is

$$q^mL_{-m}A_m+q^mA_{-m}L_m+q^mA_{-m}A_m=0\,. \tag{5.5}$$

In the limit case, (5.5) reduces to

$$\lim_{q\to1}\left(\mathbf{L}^L+\mathbf{A}^L\right)\cdot\left(\mathbf{L}^L+\mathbf{A}^L\right)=l(l+1)\,. \tag{5.6}$$

When the parameter l tends to infinity, $\frac{A^L_i}{l}$ tends to zero. For large l, the (5.5) reads:

$$q^mx^L_{-m}A^L_m+q^mA^L_{-m}x^L_m+q^m\frac{A^L_{-m}A^L_m}{\ell}=0\,. \tag{5.7}$$

A^L_i is to remain bounded as l tends to infinity. Also, in this limit x^L_i tends to $\hat{x}_i$, the unit normal to the S^2 at $\hat{\mathbf{x}}$. So in the limiting case, if A^L_i tends to a_i then $\hat{\mathbf{x}}\cdot\mathbf{a}=0$.

Now, following Ref. 5 we can introduced the q-deformed gauged Ginsparg–Wilson system as follows: we can set

$$\Gamma^{q\mu}\big(\mathbf{A}^L\big) = \frac{[2]_q \mathbf{\Sigma}^q \cdot \big(\mathbf{L}^L + \mathbf{A}^L\big) + 1}{\big|[2]_q \mathbf{\Sigma}^q \cdot \big(\mathbf{L}^L + \mathbf{A}^L\big) + 1\big|}\,. \tag{5.8}$$

It is an involutory and $*$-invariant operator:

$$\Gamma^{q\mu}\big(\mathbf{A}^L\big)^2 = 1\,, \quad \Gamma^{q\mu}\big(\mathbf{A}^L\big)^* = \Gamma^{q\mu}\big(\mathbf{A}^L\big)\,. \tag{5.9}$$

The gauged involution (5.8), reduces to (3.13) for zero $\mathbf{A}^L$. We put $\Gamma^{q\mu} = \Gamma^{q\mu}(\mathbf{A}^L = 0)$.

Also, we can define the second gauged involution as

$$\Gamma'^{\,q\mu}\big(\mathbf{A}^L\big) = \Gamma'^{\,q\mu}(0) = \Gamma'^{\,q\mu}\,. \tag{5.10}$$

We put $\Gamma' = \Gamma'\big(A^L = 0\big)$. Notice that, the operators $\mathbf{L}^{L,R}$ do not have continuum limit as their squares $l(l+1)$ diverge as l tends to infinity. In contrast, $\boldsymbol{\mathcal{L}}$ and $\mathbf{A}^L$ do have continuum limits.

It is easy to see that, up to the first order, (5.8) becomes:

$$\Gamma^{q\mu}\big(\mathbf{A}^L\big) = \frac{[2]_q \mathbf{\Sigma}^q \cdot \big(\mathbf{L}^L + \mathbf{A}^L\big) + 1}{[2l+1]_q} \tag{5.11}$$

and

$$\Gamma'^{\,q\mu} = \frac{-[2]_q \mathbf{\Sigma}^q \cdot \mathbf{L}^R + 1}{[2l+1]_q}\,. \tag{5.12}$$

In the limit case, when the parameter q tends to unit, (5.11) tends to:

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$$\lim_{q\to 1} \Gamma^{q\mu}\big(\mathbf{A}^L\big) = \frac{\sigma \cdot \big(L^L + A^L\big) + \frac{1}{2}}{l + \frac{1}{2}}\,. \tag{5.13}$$

Using (5.11) and (5.12) we can construct the following $*$-invariant operators:

$$\begin{aligned} \Gamma_1^{q\mu}\big(A^L\big) &= \frac{1}{[2]_q}\big(\Gamma^{q\mu}\big(A^L\big) + \Gamma'^{q\mu}\big)\,, \quad \big(\Gamma_1^{q\mu}\big)^* = \Gamma_1^{q\mu}\,, \\ \Gamma_2^{q\mu}\big(A^L\big) &= \frac{1}{[2]_q}\big(\Gamma^{q\mu}\big(A^L\big) - \Gamma'^{\,q\mu}\big)\,, \quad \big(\Gamma_2^{q\mu}\big)^* = \Gamma_2^{q\mu}\,. \end{aligned} \tag{5.14}$$

Now, let us define the gauged q-deformed fuzzy Dirac and chirality operators on q-deformed fuzzy sphere as

$$D_{q\mu}\big(A^L\big) = [2l+1]_q \Gamma_1^{q\mu}\big(A^L\big) = \frac{1}{[2]_q}\big([2]_q \mathbf{\Sigma}^q \cdot \big(\mathcal{L}^q + A^L\big) + 2\big)\,, \tag{5.15}$$

and for chirality operator:

$$\gamma_{q\mu}\big(A^L\big) = \Gamma_2^{q\mu}\big(A^L\big) = \frac{[2]_q \mathbf{\Sigma}^q \cdot \big(\mathbf{L}^L + L^R + \mathbf{A}^L\big)}{[2l+1]_q}\,. \tag{5.16}$$

Here, let us apply the third limit case mentioned before to (5.15) and (5.16):

$$\lim_{q\to 1,\mu\to 0} D_{q\mu}\big(A^L\big) = \sigma\cdot\big(\mathcal{L}+A^L\big)+1\,,\quad \lim_{q\to 1,\,\mu\to 0}\gamma_{q\mu}(A^L)=\sigma\cdot\mathbf{x}\,. \tag{5.17}$$

These are the correct gauged Dirac and chirality operators on commutative S^2.

6. Instanton Coupling

As we mentioned in Sec. 2, according to the Serre–Swan theorem, in noncommutative geometry, spaces are replaced with algebra of functions on them. Therefore, study of the quantum principal fibration $SU_q(2) \xrightarrow{U_q(1)} S_q^2$, are replaced with the study of noncommutative finitely generated projective module of the Hopf–Galois extension $A(S_q^2) \hookrightarrow A(SU_q(2))$. To build the projective module, let introduce $\mathbb{C}^{2t+1}$ carrying the t-representation of angular momentum of $U_q(su(2))$. Here, the algebra $U_q(su(2)) \equiv su_q(2)$ is generated by elements $T_\pm$ and T_3 satisfying the relation

$$\big[T_+,T_-\big] = \frac{T_3^2 - T_3^{-2}}{q-q^{-1}}\,,\quad T_3 J_\pm T_3^{-1} = q^\pm T_{\pm 1}\,,\quad T_3^* = T_3\,,\quad T_\pm^* = T_\mp\,, \tag{6.1}$$

where the $*$-structure is for real q. The structure of the Hopf algebra is given by

$$\triangle(T_\pm) = T_\pm\otimes T_3^{-1} + T_3\otimes T_\pm\,,\quad \triangle(T_3) = T_3\otimes T_3\,. \tag{6.2}$$

Also, let $P_{q\mu}^{(l+t)}$ be the projector coupling left angular momentum operator $\mathbf{L}^L$ with $\mathbf{T}$ to produce maximum angular momentum $l+t$. We know that the image of a projector on a free module is a projective module. Then, as $\mathrm{Mat}(2l+1)^{2t+1} = \mathrm{Mat}(2l+1)\otimes\mathbb{C}^{2t+1}$ is a free module, therefore, $P_{q\mu}^{(l+t)}\,\mathrm{Mat}(2l+1)^{2t+1}$ is the fuzzy version of $U(1)$ bundle on $S_{q\mu}^2$. Also, we can use the projector $P_{q\mu}^{(l-t)}$ to produce the projective module $P_{q\mu}^{(l-t)}\,\mathrm{Mat}(2l+1)^{2t+1}$ to introduce the least angular momentum $(l-t)$.

The q-deformed fuzzy projectors $P_{q\mu}^{l\pm t}$ corresponding to $(l\pm t)$-representations of $su_q(2)$ can be written as

$$P_{q\mu}^{(l\pm t)} = \frac{[2]_q\boldsymbol{\Sigma}^q\cdot(\mathbf{L}^L+\mathbf{T}) + [2(l\pm t)+1]_q + 1}{[2]_q[2(l\pm t)+1]_q}\,,\quad P_{q\mu}^{(l\pm t)^*} = P_{q\mu}^{l\pm t}\,, \tag{6.3}$$

$$\begin{aligned}\mathrm{Mat}(2l+1)\otimes\mathbb{C}^{2t+1} &= \big(\mathrm{Mat}(2l+1)\otimes\mathbb{C}^{2t+1}\big)P_{q\mu}^{(l+t)}\\ &\quad\oplus\big(\mathrm{Mat}(2l+1)\otimes\mathbb{C}^{2t+1}\big)P_{q\mu}^{(l-t)}\,.\end{aligned} \tag{6.4}$$

To set the q-deformed fuzzy Ginsparg–Wilson system in instanton sector, we choose the following $*$-invariant involution Γ for the highest and lowest weights $l\pm t$:

$$\begin{gathered}\Gamma_{q\mu}^\pm(\mathbf{T}) = [2]_q P_{q\mu}^{(l\pm t)} - 1 = \frac{[2]_q\boldsymbol{\Sigma}^q\cdot(\mathbf{L}^L+\mathbf{T})+1}{[2(l\pm t)+1]_q}\,,\\ \big(\Gamma_{q\mu}^\pm(\mathbf{T})\big)^2 = 1\,,\quad \Gamma_{q\mu}^{\pm^*}(\mathbf{T}) = \Gamma_{q\mu}^\pm(\mathbf{T})\,.\end{gathered} \tag{6.5}$$

It is clear that $\Gamma^{\pm}_{q\mu}(T=0) = \Gamma_{q\mu}$. On the module $(\mathrm{Mat}(2l+1)^{2t+1} \otimes \mathbb{C}^2)P^{(l\pm t)}_{q\mu}$ we have

$$\left(\mathbf{L}^L + \mathbf{T}\right)^2 = \frac{[2(l\pm t)]_q[2(l\pm t)+2]_q}{[2]_q^2}\,. \tag{6.6}$$

We choose $\Gamma'_{q\mu}$ to be the same as in (4.1). Now, we can introduce our q-deformed fuzzy Ginsparg–Wilson system in instanton sector as

$$\mathcal{A}^{\pm}_{q\mu}(T) = \left\{\Gamma^{\pm}_{q\mu}(\mathbf{T}), \Gamma'_{q\mu} :\ \Gamma^{\pm^2}_{q\mu}(\mathbf{T}) = \Gamma'^2_{q\mu} = 1,\ \Gamma^{\pm *}_{q\mu}(\mathbf{T}) = \Gamma^{\pm}_{q\mu}(\mathbf{T}),\ \Gamma'^*_{q\mu} = \Gamma'_{q\mu}\right\}. \tag{6.7}$$

To introduce the new Dirac operator, first, let compute Γ_1. The result is

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$$\Gamma^{\pm}_1(T) = \frac{[2]_q[2l+1]_q\Sigma^q\cdot(L^L+T) + [2l+1]_q - [2]_q[2(l\pm t)+1]_q\Sigma^q\cdot L^R + [2(l\pm t)+1]_q}{[2]_q[2l+1]_q[2(l\pm t)+1]_q}\,, \tag{6.8}$$

so the q-deformed fuzzy Dirac operator in instanton sector corresponding to the Ginsparg–Wilson algebra (6.7) is

$$\begin{aligned} D^{\pm}_{q\mu}(T) &= [2l+1]_q[2(l\pm t)+1]_q\Gamma^{\pm}_1(T) \\ &= [2l+1]_q\Sigma^q\cdot(L^T+T) - [2(l\pm t)+1]_q\Sigma^q\cdot L^R \\ &\quad + \frac{[2l+1]_q + [2(l\pm t)+1]_q}{[2]_q}\,. \end{aligned} \tag{6.9}$$

It is obvious that the Dirac operator (6.9) is $*$-invariant:

$$D^{(l\pm t)^*}_{q\mu}(T) = D^{(l\pm t)}_{q\mu}(T)\,. \tag{6.10}$$

In the limit case the Dirac operator (6.9) reduces to

$$\lim_{q\to 1,\mu\to 0} D_{q\mu}(T) = \sigma\cdot(\mathcal{L}+T) + 1\,, \tag{6.11}$$

which we expect from commutative Dirac operator in instanton sector.

7. Gauging the *q*-Deformed Fuzzy Dirac Operator in Instanton Sector

The derivation $\mathcal{L}_i$ dose not commute with the projectors $P^{l\pm t}_{q\mu}$ and then has no action on the modules $\mathrm{Mat}(2l+1)P^{(l\pm t)}_{q\mu}$. But $J_i = q^{\frac{-1}{2}}\mathcal{L}_i + \frac{1}{[2]_q}T_i$ does commute with $P^{l\pm t}_{q\mu}$. Here, J_i has been considered as the total angular momentum.

Now, we need to gauge J_i. When $T=0$, the gauge fields A_i were function of L^L_i. Here, we consider A^L_i to be a functions of $\mathbf{L}^L + \mathbf{T}$, because A^L_i does not commute with $P^{l\pm t}_{q\mu}$. Let us introduce the covariant derivative as

$$\nabla_i = J_i + A^L_i\,. \tag{7.1}$$

In this case, the limiting transversality of $\mathbf{L}^L + \mathbf{T}$ can be guaranteed by imposing the condition

$$\begin{aligned}&\big(\mathbf{L}^L + \mathbf{A}^L + \mathbf{T}\big) \cdot \big(\mathbf{L}^L + \mathbf{A}^L + \mathbf{T}\big)\\&\quad = \big(\mathbf{L}^L + \mathbf{T}\big) \cdot \big(\mathbf{L}^L + \mathbf{T}\big) = \sum q^m \big(L^L + T\big)_{-m} \big(L^L + T\big)_m\\&\quad = \frac{[2(l \pm t)]_q [2(l \pm t) + 2]_q}{[2]_q^2}\,. \end{aligned} \tag{7.2}$$

In the limit case we have

$$\lim_{q \to 1} \big(\mathbf{L}^L + \mathbf{A}^L + T\big) \cdot \big(\mathbf{L}^L + \mathbf{A}^L + T\big) = (l \pm t)((l \pm t) + 1)\,. \tag{7.3}$$

The expansion of (7.2) is

$$q^m (L+T)_{-m} A_m + q^m A_{-m} (L+T)_m + q^m A_{-m} A_m = 0\,. \tag{7.4}$$

When the parameter l tends to infinity, $\frac{A_i^L}{l}$ tends to zero. For large l, (7.4) is

$$q^m x_{-m}^L A_m^L + q^m A_{-m}^L x_m^L + q^m \frac{A_{-m}^L A_m^L}{l} = 0\,. \tag{7.5}$$

A_i^L is to remain bounded as l tends to infinity. Also, in this limit x_i^L tends to $\hat{x}_i$, the unit normal to the S^2 at $\hat{\mathbf{x}}$. So in the limiting case, if A_i^L tends to a_i, then $\hat{\mathbf{x}} \cdot \mathbf{a} = 0$.

Now, we can construct the gauged q-deformed fuzzy Ginsparg–Wilson system in instanton sector and its corresponding Dirac and chirality operators as follows:

$$\begin{aligned}\mathcal{A}_{q\mu}^{\pm}(T, A^L) = \Big\{ &\Gamma_{q\mu}^{\pm}\big(\mathbf{T}, \mathbf{A^L}\big),\ \Gamma'_{q\mu} :\ {\Gamma_{q\mu}^{\pm}}^2\big(\mathbf{T}, \mathbf{A^L}\big) = \Gamma'^2_{q\mu} = 1,\\ &\Gamma_{q\mu}^{\pm *}\big(\mathbf{T}, \mathbf{A^L}\big) = \Gamma_{q\mu}^{\pm}(\mathbf{T}),\ \Gamma'^*_{q\mu} = \Gamma'_{q\mu} \Big\}\,. \end{aligned} \tag{7.6}$$

We introduce the involutory $*$-invariant generators of the Ginsparg–Wilson system as

$$\begin{aligned}\Gamma^{q\mu}\big(T, \mathbf{A}^L\big) &= \frac{[2]_q \mathbf{\Sigma}^q \cdot \big(\mathbf{L}^L + \mathbf{T} + \mathbf{A}^L\big) + 1}{|[2]_q \mathbf{\Sigma} \cdot (\mathbf{L}^L + \mathbf{T} + \mathbf{A}^L) + 1|}\,,\\ \Gamma'^{q\mu} &= \frac{-[2]_q \mathbf{\Sigma}^q \cdot \mathbf{L}^R + 1}{|-[2]_q \mathbf{\Sigma}^q \cdot (\mathbf{L}^R) + 1|}\,. \end{aligned} \tag{7.7}$$

Now, up to the first order (7.7) becomes:

$$\begin{aligned}\Gamma^{q\mu}\big(T, \mathbf{A}^L\big) &= \frac{[2]_q \mathbf{\Sigma}^q \cdot \big(\mathbf{L}^L + \mathbf{T} + \mathbf{A}^L\big) + 1}{[2(l \pm t) + 1]_q}\,,\\ \Gamma'^{q\mu} &= \frac{-[2]_q \mathbf{\Sigma}^q \cdot \mathbf{L}^R + 1}{[2l + 1]_q}\,. \end{aligned} \tag{7.8}$$

To introduce the new Dirac operator, first, let compute Γ_1. The result is

$$\Gamma_1^{\pm}(T, A^L) = \frac{[2]_q[2l+1]_q\Sigma^q\cdot\left(L^L+T+A^L\right)+[2l+1]_q-[2]_q[2(l\pm t)+1]_q\Sigma^q\cdot L^R+[2(l\pm t)+1]_q}{[2]_q[2l+1]_q[2(l\pm t)+1]_q}, \tag{7.9}$$

so the q-deformed fuzzy Dirac operator in instanton sector corresponding to the Ginsparg–Wilson algebra (7.6) is

$$\begin{aligned} D_{q\mu}^{\pm}(T, A^L) &= [2l+1]_q[2(l\pm t)+1]_q\Gamma_1^{\pm}(T, A^L) \\ &= [2l+1]_q\Sigma^q\cdot\left(L^T+T+A^L\right) - [2(l\pm t)+1]_q\Sigma^q\cdot L^R \\ &\quad + \frac{[2l+1]_q + [2(l\pm t)+1]_q}{[2]_q}\,. \end{aligned} \tag{7.10}$$

It is obvious that this Dirac operator is $*$-invariant:

$$D_{q\mu}^{(l\pm t)^*}(T, A^L) = D_{q\mu}^{(l\pm t)}(T, A^L)\,. \tag{7.11}$$

In the limit case the Dirac operator (7.10) reduces to

$$\lim_{q\to1,\mu\to0} D_{q\mu}(T, A^L) = \sigma\cdot\left(\mathcal{L}+T+A^L\right)+1\,, \tag{7.12}$$

which we expect from commutative gauged Dirac operator in instanton sector.

8. Conclusion

In this paper, using the q-deformed projectors and idempotents of the finitely generated q-deformed projective $A(S_{q\mu}^2)$-module, we have constructed the generators of the q-deformed gauged fuzzy Ginsparg–Wilson algebra in instanton sector. We have also constructed q-deformed gauged fuzzy Dirac operator in instanton sector using the q-deformed fuzzy GW algebra. The importance of this Dirac operator is that it has correct commutative limit.

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