GENERALIZED VOLTERRA TYPE INTEGRAL OPERATORS ON BERGMAN SPACES WITH LOGARITHMIC WEIGHTS

Mahdi MAHBOOBI[1], Ali EBADIAN[2] and Shahram NAJAFZADEH[2]

Fix $-1 < \gamma < \infty$ and $\delta \leqslant 0$. The logarithmic weight $\omega_{\gamma,\delta}(z)$ is defined by

$$\omega_{\gamma,\delta}(z) = \left(\log \frac{1}{|z|}\right)^{\gamma} \left[\log\left(1 - \frac{1}{\log|z|}\right)\right]^{\delta}.$$

In this paper, using $\omega_{\gamma,\delta}$-Carleson measure and modified Nevanlinna counting function, we study when a generalized Volterra type integral operator acting on Bergman spaces of logarithmic weights is bounded.

Keywords: admissible weight, Bergman space with logarithmic weight, generalized Volterra type integral operator, modified Nevanlinna counting function, $\omega_{\gamma,\delta}$-Carlesson measure.

1. Introduction

Let $\mathbb{D}$ denotes the open unit disk in the complex plan $\mathbb{C}$. We will use the notation $\mathcal{H}(\mathbb{D})$ for the class of all analytic functions on $\mathbb{D}$. The generalized Volterra type integral operator induced by the function $g \in \mathcal{H}(\mathbb{D})$ and the self-map φ of $\mathbb{D}$, is defined as follows:

$$J_g^{\varphi} : \mathcal{H}(\mathbb{D}) \to \mathcal{H}(\mathbb{D}), \quad h \mapsto \int_0^z h\left(\varphi(\zeta)\right) g'(\zeta) d\zeta, \quad (z \in \mathbb{D}).$$

Recall that (see, for example [2]) a normal weight ω that is a continuous function such that:

(i) ω is a radial weight, that is $\omega(z) = \omega(|z|)$ for every z;

(ii) there exists $t > s > 0$ such that

$$\frac{\omega(r)}{(1-r)^s} \searrow 0, \quad \frac{\omega(r)}{(1-r)^t} \nearrow \infty,$$

as $r \to 1^-$.

We say that ω is an admissible weight if it is non-increasing and $\omega(r)(1-r)^{-(1+\eta)}$ is non-decreasing for some $\eta > 0$.

A word on notation: The notation $U(z) \lesssim V(z)$ (or respectively $U(z) \gtrsim V(z)$) means that there is a constant C such that $U(z) \leqslant CV(z)$ (or respectively $V(z) \leqslant CU(z)$) holds for all z in the set in question. We write $U(z) \approx V(z)$ if both $U(z) \lesssim V(z)$ and $V(z) \lesssim U(z)$ hold (see for example [2]).

[1]Department of Mathematics, Payame Noor University, 19395-3697, Tehran, Iran, e-mail:mahboobimath@gmail.com

[2]Department of Mathematics, Faculty of science, Urmia university, Urmia, Iran, e-mail:ebadian.ali@gmail.com

[3]Department of Mathematics, Payame Noor University, 19395-3697, Tehran, Iran, e-mail:Najafzadeh1234@yahoo.ie

Let $0 < p < \infty$ and ω be a normal weight function on $\mathbb{D}$. Then the space $\mathcal{A}(p,\omega)$ is defined as follows:

$$\mathcal{A}(p,\omega) = \left\{ f \in \mathcal{H}(\mathbb{D});\ \|f\|^p_{\mathcal{A}(p,\omega)} = \int_{\mathbb{D}} |f(z)|^p \frac{\omega(|z|)}{1-|z|}\, dA(z) < \infty \right\},$$

where $dA(z)$ is the area measure on $\mathbb{D}$ normalized so that area of $\mathbb{D}$ is 1.

For $1 \le p < \infty$, $\mathcal{A}(p,\omega)$ is a Banach space equipped with the norm $\|\cdot\|_{\mathcal{A}(p,\omega)}$. When $0 < p < 1$, $\|\cdot\|_{\mathcal{A}(p,\omega)}$ is a quasinorm on $\mathcal{A}(p,\omega)$ and $\mathcal{A}(p,\omega)$ is a Fréchet space, but not a Banach space. By a Fréchet space, we mean a topological vector space X if it is a Hausdorff and complete space with respect to the family of semi-norms such that its topology may be induced by a countable family of semi-norms. Moreover, the following asymptotic relation, for all $z \in \mathbb{D}$ and $n \in \mathbb{N}$, holds

$$\|f\|_{\mathcal{A}(p,\omega)} \approx \sum_{j=0}^{n-1} \left| f^{(j)}(0) \right| + \left(\int_{\mathbb{D}} \left| f^{(n)}(z) \right|^p \left(1-|z|^2\right)^{pn} \frac{\omega(|z|)}{1-|z|}\, dA(z) \right)^{1/p}. \tag{1.1}$$

This relation is well-known and can be found for standard power weights in [1], Theorem 6 and 7 (for the n-dimensional case, we refer to [2]).

Let $-1 < \gamma < \infty$, $\delta \leqslant 0$ and $0 < p < \infty$. The logarithmic weighted Bergman space $\mathcal{A}^p_{\omega_{\gamma,\delta}}$ consists of all holomorphic functions on $\mathbb{D}$ such that

$$\|f\|^p_{\mathcal{A}^p_{\omega_{\gamma,\delta}}} = \int_{\mathbb{D}} |f(z)|^p\, \omega_{\gamma,\delta}(z)\, dA(z) < \infty,$$

where the logarithmic weight is defined by

$$\omega_{\gamma,\delta}(z) = \left(\log \frac{1}{|z|}\right)^{\gamma} \left[\log\left(1 - \frac{1}{\log|z|}\right)\right]^{\delta}. \tag{1.2}$$

When $\gamma = 0$, $\delta = 0$, this space becomes the Bergman space A^p and when $\delta = 0$, it is the weighted Bergman space A^p_γ.

Let φ be a holomorphic self-map of $\mathbb{D}$, $0 \le r < 1$, $0 \le \gamma < \infty$, $\delta \le 0$ and $a \in \mathbb{D} \setminus \{\varphi(0)\}$. An important ingredient, in our study, is the use of the modified Nevanlinna counting function associated with φ and defined as follows in [5]

$$N_{\varphi,\gamma,\delta}(r,a) = \sum_{z_j(a) \in \varphi^{-1}(a)} \omega_{\gamma,\delta}\left(\frac{z_j(a)}{r}\right) \tag{1.3}$$

with $|z_j(a)| < r$, counting multiplicities, and

$$N_{\varphi,\gamma,\delta}(a) = N_{\varphi,\gamma,\delta}(1,a) = \sum_{z_j(a) \in \varphi^{-1}(a)} \omega_{\gamma,\delta}\left(z_j(a)\right). \tag{1.4}$$

We consider $N_{\varphi,\gamma,\delta}(r,a)$ to be defined on $\mathbb{D} \setminus \{\varphi(0)\}$ and $N_{\varphi,\gamma,\delta}(r,a) = 0$ if a is not in $\varphi(r\mathbb{D})$ where $r\mathbb{D} = \{z \in \mathbb{D} : |z| < r\}$. When $\delta = 0$, we denote, as introduced by Shapiro ([7]),

$$N_{\varphi,\gamma}(r,a) = \sum_{z \in \varphi^{-1}(a), |z| < r} \left(\log \frac{r}{|z|}\right)^{\gamma}$$

and

$$N_{\varphi,\gamma}(a) = N_{\varphi,\gamma}(1,a) = \sum_{z \in \varphi^{-1}(a)} \left(\log \frac{1}{|z|}\right)^{\gamma}.$$

For any $-\infty < \alpha < +\infty$ we consider the positive measure $dA_\alpha(z) = (1-|z|^2)^\alpha dA(z)$. It is easy to see that dA_α is finite if and only if $\alpha > -1$. When $\alpha > -1$, we normalize dA_α

so that it is a probability measure. Bergman spaces with standard weights are defined as follows:

$$\mathcal{A}_\alpha^p(\mathbb{D}) = \mathcal{H}(\mathbb{D}) \cap L^p(\mathbb{D}, dA_\alpha),$$

where $p > 0$ and $\alpha > -1$. Shapiro [7] expressed the essential norm of the composition operator on $\mathcal{A}_\alpha^2(\mathbb{D})$ in terms of the generalized Nevanlinna counting function. Also, E. G. Know and J. Lee [4] studied the similar argument for the composition operators on Bergman spaces of logarithmic weights in terms of the modified Nevanlinna counting function. Furthermore, F. Pérez-González, J. Rättyä and D. Vukotić [6] gave several quantities for the essential norm $\|C_\varphi\|$, where the essential norm $\|C_\varphi\|$ of the bounded operator C_φ is its distance (in the operator norm) from compact operators, that is: $\|C_\varphi\| = \inf_K \|C_\varphi - K\|$ where the infimum is taken over all admissible compact operators.

All the functions $f,\ g$ and h under consideration are assumed to be holomorphic on $\mathbb{D}$. Moreover, φ always denotes a holomorphic self-map of $\mathbb{D}$.

2. Preliminaries

We give the following definitions and auxiliary results that will be required in the later section.

Definition 2.1. *Let $-1 < \gamma < \infty$ and $\delta \leqslant 0$. We say that the positive Borel measure μ is a $\omega_{\gamma,\delta}$-Carleson measure if there is a constant $C > 0$ such that for all $f \in \mathcal{A}^p_{\omega_{\gamma,\delta}}$,*

$$\int_{\mathbb{D}} |f(z)|^p \ d\mu(z) \leqslant C \, \|f\|^p_{\mathcal{A}^p_{\omega_{\gamma,\delta}}}.$$

Here, we define the following concepts.

Definition 2.2. *Let $h, f \in \mathcal{H}(\mathbb{D})$. If $h(z) = O\left(f(z)\right),\ |z| \to 1^-$ and $f(z) = O\left(h(z)\right),\ |z| \to 1^-$ simultaneous, then we denote this concept by $O\left(f(z)\right) = O\left(h(z)\right),\ |z| \to 1^-$. Namely, there exists $r_0 \in [0, 1)$ such that $h(z) \approx f(z)$ for $r_0 \leq |z| < 1$.*

Now, we quote several lemmas which will be used in the proofs of the main results in this paper.

Lemma 2.1. ([4], Lemma 3.1)

$$\log\left(1 - \frac{1}{\log x}\right) \approx \log\frac{1}{1-x}, \quad \frac{1}{2} \leq x < 1.$$

Lemma 2.2. ([5], Lemma 2.3) *If g is a non-negative measurable function on $\mathbb{D}$, then*

$$\int_{\mathbb{D}} (g \circ \varphi)(z)|\varphi'(z)|^2 \omega(z) dA(z) = \int_{\mathbb{D}} g(u) N_{\varphi,\gamma,\delta}(u) \ dA(u).$$

Lemma 2.3. ([4], Lemma 3.3) *Let $1 \leq \gamma < \infty,\ \alpha, \delta \leq 0$ and $0 < m, t < \infty$ with $m - t > -\alpha$. Then*

$$N_{\varphi,\gamma,\delta}(a) = O\left(\omega_{t,\alpha}(a)\right) \quad (|a| \to 1^-) \tag{2.1}$$

if and only if

$$\sup_{a\in\mathbb{D}} \frac{1}{\omega_{t,\alpha}(a)} \int_{\mathbb{D}} \frac{(1-|a|^2)^m}{|1-\overline{a}z|^{m+2}} N_{\varphi,\gamma,\delta}(z) \ dA(z) < \infty. \tag{2.2}$$

In particular,

$$\limsup_{|a|\to 1} \frac{N_{\varphi,\gamma,\delta}(a)}{\omega_{t,\alpha}(a)} \approx \limsup_{|a|\to 1} \frac{1}{\omega_{t,\alpha}(a)} \int_{\mathbb{D}} \frac{(1-|a|^2)^m}{|1-\overline{a}z|^{m+2}} N_{\varphi,\gamma,\delta}(z) \ dA(z). \tag{2.3}$$

Lemma 2.4. ([5], Lemma 3.2) *For a fixed* $r_0 \in [0,1)$

$$\|f\|^p_{\mathcal{A}^p_{\omega_{\gamma,\delta}}} \approx \int_{\mathbb{D}\setminus r_0\mathbb{D}} |f(z)|^p\, \omega_{\gamma,\delta}(z)\ dA(z).$$

Lemma 2.5. ([3], Lemma 2.4) *Suppose that* ω *is an admissible weight function and* $\eta > 0$. *Then for any* $a \in \mathbb{D}$, *we have*

$$\int_{\mathbb{D}} \frac{\omega(z)}{|1-\bar{a}z|^{4+2\eta}}\ dA(z) \approx \frac{\omega(a)}{(1-|a|^2)^{2+2\eta}}.$$

3. Main Results

In this section, we characterize the boundedness of the generalized Volterra type integral operators on Bergman spaces with logarithmic weights.

Proposition 3.1. *Let* $0 < p < \infty$, $-1 < \gamma < \infty$ *and* $\delta \leqslant 0$. *For a holomorphic function* $h \in \mathcal{H}(\mathbb{D})$, *such that* $h(0) = 0$,

$$\|h\|^p_{\mathcal{A}_{\omega_{\gamma,\delta}}} \approx \int_{\mathbb{D}} |h'(z)|^p\, \omega_{(\gamma+1)/p+p-1,\delta/p}(z)\ dA(z), \tag{3.1}$$

Proof. Step 1. By the method that is used, in the proof of Lemma 3.2, in [5] we can see that for a fixed $r_0 \in [0,1)$,

$$\|f\|^p_{\mathcal{A}^p_{\omega_{\gamma,\delta}}} = \int_{\mathbb{D}\setminus r_0\mathbb{D}} |f(z)|^p\, \frac{\omega(|z|)}{1-|z|}\ dA(z). \tag{3.2}$$

for a normal weight function ω on $\mathbb{D}$.

Step 2. We put

$$\vartheta_{\gamma,\delta}(z) = (1-|z|)^\gamma \left[\log\left(\frac{1}{1-|z|}\right)\right]^\delta, \quad (z \in \mathbb{D}). \tag{3.3}$$

Since, $(1-|z|) \approx \left[\log \dfrac{1}{|z|}\right]$ and $\log\left(1 - \frac{1}{\log|z|}\right) \approx \log \frac{1}{1-|z|}$ (Lemma 2.1), for $1/2 \leq |z| < 1$, we have

$$\vartheta_{\gamma,\delta}(z) \approx \omega_{\gamma,\delta}(z) \tag{3.4}$$

for $1/2 \leq |z| < 1$, where $\omega_{\gamma,\delta}(z)$ is the weight defined in (1.2). Similar to the proof of Lemma 3.2 in [5], for a fixed $r_0 \in [1/2,1)$, we can see

$$\begin{aligned} \int_{r_0\mathbb{D}} |f(z)|^p\, \omega_{\gamma,\delta}(z)\ dA(z) &\leq \int_{\mathbb{D}\setminus r_0\mathbb{D}} |f(z)|^p\, \omega_{\gamma,\delta}(z)\ dA(z) \\ \int_{r_0\mathbb{D}} |f(z)|^p\, \vartheta_{\gamma,\delta}(z)\ dA(z) &\leq \int_{\mathbb{D}\setminus r_0\mathbb{D}} |f(z)|^p\, \vartheta_{\gamma,\delta}(z)\ dA(z), \end{aligned} \tag{3.5}$$

and by relation (3.4) we have

$$\int_{\mathbb{D}\setminus r_0\mathbb{D}} |f(z)|^p\, \omega_{\gamma,\delta}(z)\ dA(z) \approx \int_{\mathbb{D}\setminus r_0\mathbb{D}} |f(z)|^p\, \vartheta_{\gamma,\delta}(z)\ dA(z). \tag{3.6}$$

Now, by Lemma 2.4, relations (3.5) and (3.6), we get

$$\begin{aligned} \|f\|^p_{\mathcal{A}_{\omega_{\gamma,\delta}}} &\lesssim \int_{\mathbb{D}\setminus r_0\mathbb{D}} |f(z)|^p\, \omega_{\gamma,\delta}(z)\ dA(z) \\ &\lesssim \int_{\mathbb{D}\setminus r_0\mathbb{D}} |f(z)|^p\, \vartheta_{\gamma,\delta}(z)\ dA(z) \\ &\leq \int_{\mathbb{D}} |f(z)|^p\, \vartheta_{\gamma,\delta}(z)\ dA(z). \end{aligned}$$

So, we can easily see that

$$\|f\|^p_{\mathcal{A}_{\omega_{\gamma,\delta}}} \approx \int_{\mathbb{D}} |f(z)|^p \, \vartheta_{\gamma,\delta}(z) \; dA(z). \tag{3.7}$$

Step 3. Putting

$$\omega(z) = \left(\vartheta_{\gamma+1,\delta}(z)\right)^{1/p}.$$

We can easily see that the weight $\omega(z)$ defined in the above is normal. So, by (3.2), equation (3.6) and Lemma 2.4,

$$\begin{aligned} \|h\|^p_{\mathcal{A}_{(p,\omega)}} &\approx \int_{\mathbb{D}\setminus r_0\mathbb{D}} |h(z)|^p \, \frac{(1-|z|)^{\gamma+1}\left(\log \dfrac{1}{1-|z|}\right)^{\delta}}{(1-|z|)} \; dA(z) \\ &\approx \int_{\mathbb{D}} |h(z)|^p \, \vartheta_{\gamma,\delta}(z) \; dA(z) \\ &\approx \|h\|^p_{\mathcal{A}^p_{\omega_{\gamma,\delta}}}. \end{aligned} \tag{3.8}$$

If we put $n = 1$ in the relation (1.1), since $h(0) = 0$, we have

$$\begin{aligned} \|h\|_{\mathcal{A}_{(p,\omega)}} &\approx \left(\int_{\mathbb{D}\setminus r_0\mathbb{D}} |h'(z)|^p \left(1-|z|^2\right)^p \frac{\omega(|z|)}{1-|z|} \; dA(z)\right)^{1/p} \\ &\approx \left(\int_{\mathbb{D}} |h'(z)|^p \, \omega_{(\gamma+1)/p+p-1,\delta/p}(z) \; dA(z)\right)^{1/p}. \end{aligned} \tag{3.9}$$

By relations (3.8) and (3.9) the proof is complete. □

Lemma 3.1. *Let $0 < p < \infty$, $0 \leqslant \gamma < \infty$ and $\delta \leqslant 0$. The weight $\vartheta_{\gamma,\delta}$ is an admissible weight.*

Proof. Let $r_1 < r_2$. Since $\delta \leqslant 0$,

$$\left(\log \frac{1}{1-r_1}\right)^{\delta} > \left(\log \frac{1}{1-r_2}\right)^{\delta}.$$

We know

$$(1-r_1)^{\gamma} > (1-r_2)^{\gamma},$$

thus $\vartheta_{\gamma,\delta}(r)$ is non-increasing. We have

$$\frac{1-r_1}{1-r_2} > 1,$$

so,

$$\left(\frac{1-r_1}{1-r_2}\right)^{1/\delta} < 1.$$

Thus,

$$\frac{\log(1-r_1)}{\log(1-r_2)} > 1 > \left(\frac{1-r_1}{1-r_2}\right)^{1/\delta}.$$

Since $\delta \leqslant 0$,

$$\left(\frac{\log(1-r_1)}{\log(1-r_2)}\right)^{\delta} < \frac{1-r_1}{1-r_2}.$$

Therefore

$$\frac{\left(\log \frac{1}{1-r_1}\right)^{\delta}}{\left(\log \frac{1}{1-r_2}\right)^{\delta}} < \frac{1-r_1}{1-r_2}$$

and then

$$\left(\log \frac{1}{1-r_1}\right)^{\delta} (1-r_1)^{-1} < \left(\log \frac{1}{1-r_2}\right)^{\delta} (1-r_2)^{-1}.$$

So it is enough we put $\eta = \gamma$, when $\gamma \neq 0$ in the definition of the admissible weight. If $\gamma = 0$, then

$$\left(\frac{1-r_1}{1-r_2}\right)^{2/\delta} < 1.$$

Therefore,

$$\left(\log \frac{1}{1-r_1}\right)^{\delta} (1-r_1)^{-2} < \left(\log \frac{1}{1-r_2}\right)^{\delta} (1-r_2)^{-2}.$$

In this case, we put $\eta = 1$, in the definition of the admissible weight, then the proof is complete. □

Before we represent the fundamental theorem of this article we let

$$d\mu_N(z) = N_{\varphi,\beta,\sigma}(z) dA(z),$$

where $\beta = (\gamma+1)/p + p - 1$ and $\sigma = \delta/p$.

Theorem 3.1. *Suppose that $1 \leqslant p < \infty$, $0 < \gamma < \infty$ and $\delta \leqslant 0$. Also, let $g \in \mathcal{H}(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$ such that $O\left(|\varphi'(z)|^2\right) = O\left(|g'(z)|^p\right)$, $|z| \to 1^-$. Then, the operator J_g^{φ} is bounded on the Bergman spaces of logarithmic weights if and only if μ_N is a $\omega_{\gamma,\delta}$-Carleson measure.*

Proof. Let J_g^{φ} be the bounded operator on $\mathcal{A}^p_{\omega_{\gamma,\delta}}$. So, there is a constant $C > 0$ such that

$$\| J_g^{\varphi} f \|_{\mathcal{A}^p_{\omega_{\gamma,\delta}}} \leqslant C \| f \|_{\mathcal{A}^p_{\omega_{\gamma,\delta}}},$$

for each $f \in \mathcal{A}^p_{\omega_{\gamma,\delta}}$. Since, $J_g^{\varphi}(f)(0) = 0$, by Proposition 3.1 we get

$$\begin{aligned} C \| f \|^p_{\mathcal{A}^p_{\omega_{\gamma,\delta}}} &\geqslant \int_{\mathbb{D}} \left| \left(J_g^{\varphi}(f)\right)'(z) \right|^p \omega_{\beta,\sigma}(z)\, dA(z) \\ &= \int_{\mathbb{D}} |f \circ \varphi(z)|^p \, |g'(z)|^p \, \omega_{\beta,\sigma}(z)\, dA(z). \end{aligned}$$

Since $|\varphi'(z)|^2 = O\left(|g'(z)|^p\right)$, $|z| \to 1$, there exists $r_0 \in (0,1)$ such that

$$|\varphi'(z)|^2 \lesssim |g'(z)|^p, \quad (r_0 \leqslant |z| < 1).$$

Therefore,

$$\| f \|^p_{\mathcal{A}^p_{\omega_{\gamma,\delta}}} \gtrsim \int_{\mathbb{D}\setminus r_0\mathbb{D}} |f \circ \varphi(z)|^p \, |\varphi'(z)|^2 \, \omega_{\beta,\sigma}(z)\, dA(z).$$

By the method that is used in the proof of Lemma 3.2 in [5], we can easily see that for a fixed $r_0 \in [0,1)$,

$$\int_{\mathbb{D}\setminus r_0\mathbb{D}} |f \circ \varphi(z)|^p \, |\varphi'(z)|^2 \, \omega_{\beta,\sigma}(z)\, dA(z) \approx \int_{\mathbb{D}} |f \circ \varphi(z)|^p \, |\varphi'(z)|^2 \, \omega_{\beta,\sigma}(z)\, dA(z).$$

Lemma 2.2 implies that

$$\begin{aligned}\| f \|^p_{\mathcal{A}^p_{\omega_{\gamma,\delta}}} &\gtrsim \int_{\mathbb{D}} |f(z)|^p \, N_{\varphi,\beta,\sigma}(z) \; dA(z) \\ &= \int_{\mathbb{D}} |f(z)|^p \;\; d\mu_N(z).\end{aligned}$$

Thus, by Definition 2.1, we get the measure $\mu_N(z)$ is an $\omega_{\gamma,\delta}$-Carleson measure.

Now, suppose that $\mu_N(z)$ is an $\omega_{\gamma,\delta}$-Carleson measure. For each $a \in \mathbb{D}$ we consider a function $k_a(z)$ defined as follow

$$k_a(z) = \frac{\left(1-|a|^2\right)^{m/p}}{|1-\bar{a}z|^{(m+2)/p}}, \quad (z \in \mathbb{D}).$$

By using relation (3.7), we get

$$\begin{aligned}\| k_a \|^p_{\mathcal{A}^p_{\omega_{\gamma,\delta}}} &\approx \int_{\mathbb{D}} |k_a(z)|^p \, \vartheta_{\gamma,\delta}(z) \; dA(z) \\ &= \int_{\mathbb{D}} \frac{\left(1-|a|^2\right)^m}{|1-\bar{a}z|^{m+2}} \vartheta_{\gamma,\delta}(z) \; dA(z).\end{aligned}$$

By Lemma 3.1 , $\vartheta_{\gamma,\delta}(z)$ is an admissible weight and so, Lemma 2.5 implies that

$$\| k_a \|^p_{\mathcal{A}^p_{\omega_{\gamma,\delta}}} \approx \vartheta_{\gamma,\delta}(a), \quad (a \in \mathbb{D}).$$

Hence, $k_a \in \mathcal{A}^p_{\omega_{\gamma,\delta}}$. By Definition 2.1, there is a constant $C > 0$ such that

$$\begin{aligned}\int_{\mathbb{D}} |k_a(z)|^p \;\; d\mu_N(z) &\leqslant C \, \| k_a \|^p_{\mathcal{A}^p_{\omega_{\gamma,\delta}}} \\ &\lesssim \vartheta_{\gamma,\delta}(a), \quad (a \in \mathbb{D}).\end{aligned}$$

If $1/2 \leqslant |a| < 1$, then $\vartheta_{\gamma,\delta}(a) \approx \omega_{\gamma,\delta}(a)$, this implies that

$$\sup_{1/2 \leqslant |a| < 1} \frac{1}{\omega_{\gamma,\delta}(a)} \int_{\mathbb{D}} \frac{\left(1-|a|^2\right)^m}{|1-\bar{a}z|^{m+2}} N_{\varphi,\beta,\sigma}(z) \; dA(z) < \infty. \tag{3.10}$$

By the relation (6), in the proof of Lemma 3.3 in [4], we have

$$\sup_{|a| < 1/2} \frac{1}{\omega_{\gamma,\delta}(a)} \int_{\mathbb{D}} \frac{\left(1-|a|^2\right)^m}{|1-\bar{a}z|^{m+2}} N_{\varphi,\beta,\sigma}(z) \; dA(z) < \infty. \tag{3.11}$$

So, equations (3.10) and (3.11) imply

$$\sup_{a \in \mathbb{D}} \frac{1}{\omega_{\gamma,\delta}(a)} \int_{\mathbb{D}} \frac{\left(1-|a|^2\right)^m}{|1-\bar{a}z|^{m+2}} N_{\varphi,\beta,\sigma}(z) \; dA(z) < \infty.$$

Therefore, by Lemma 2.3, we obtain

$$N_{\varphi,\beta,\sigma}(a) = O\left(\omega_{\gamma,\delta}(a)\right), \quad |a| \to 1^-. \tag{3.12}$$

Let $h \in \mathcal{A}^p_{\omega_{\gamma,\delta}}$. Then, by Proposition 3.1,

$$\begin{aligned}\|J_g^{\varphi}(h)\|^p_{\mathcal{A}^p_{\omega_{\gamma,\delta}}} &\approx \int_{\mathbb{D}} \left|\left(J_g^{\varphi}(h)(z)\right)'\right|^p \omega_{\beta,\sigma}(z) \; dA(z) \\ &= \int_{\mathbb{D}} |h \circ \varphi(z)|^p \, |g'(z)|^p \, \omega_{\beta,\sigma}(z) \; dA(z).\end{aligned}$$

We can see, for a fixed $r_0 \in (0,1)$,

$$\int_{\mathbb{D}} |h \circ \varphi(z)|^p \, |g'(z)|^p \, \omega_{\beta,\sigma}(z) \; dA(z) \approx \int_{\mathbb{D} \setminus r_0 \mathbb{D}} |h \circ \varphi(z)|^p \, |g'(z)|^p \, \omega_{\beta,\sigma}(z) \; dA(z).$$

Since $|g'(z)|^p = O\left(|\varphi'(z)|^2\right)$, there exists $r_0 \in (0,1)$, such that

$$|g'(z)|^p \lesssim |\varphi'(z)|^2, \quad (r_0 \leq |z| < 1).$$

So, by Lemma 2.2,

$$\begin{aligned} \|J_g^{\varphi}(h)\|^p_{\mathcal{A}^p_{\omega_{\gamma,\delta}}} &\lesssim \int_{\mathbb{D}\setminus r_0\mathbb{D}} |h\circ\varphi(z)|^p \left|\varphi'(z)\right|^2 \omega_{\beta,\sigma}(z)\; dA(z) \\ &\lesssim \int_{\mathbb{D}\setminus r_0\mathbb{D}} |h(z)|^p\, N_{\varphi,\beta,\sigma}(z)\; dA(z). \end{aligned}$$

Since the equation (3.12) is satisfied,

$$\|J_g^{\varphi}(h)\|^p_{\mathcal{A}^p_{\omega_{\gamma,\delta}}} \lesssim \|h\|^p_{\mathcal{A}^p_{\omega_{\gamma,\delta}}}.$$

Therefore, $\|J_g^{\varphi}\| \lesssim 1$ and this completes the proof. □

4. **Conclusion**

In this paper, we investigated a necessary and sufficient condition for boundedness of generalized Volterra type integral operator related to Carleson measures on Bergman spaces of logarithmic weights. Investigating this question on other Banach spaces such as Bloch spaces could be topics for our next task.

REFERENCES

[1] *T. M. Flett*, The daul of an inequality of Hardy and Littlewood and some related inequalities, J. Math. Anal. Appl., **vol. 38**, 1972, pp. 746–765.

[2] *Z. J. Hu*, Extended cesáro operators on mixed normed spaces, Proc. Amer. Math. Soc., **vol. 131**, 2003, pp. 2171–2179.

[3] *K. Kellay and P. Lefèvre*, Compact composition operators on weighted Hilbert spaces of analytic functions, J. Math. Anal. Appl., **vol. 386**, 2012, pp. 718–727.

[4] *E. G. Kwon and J. Lee*, Essential norm of the composition operators between Bergman spaces of logarithmic weights, Bull. Korean Math. Soc., **vol. 54**, no. 1, 2017, pp. 187–198.

[5] *E. G. Kwon and J. Lee*, Composition operator between Bergman spaces of logarithmic weights, Internat. J. Math., **vol. 26**, no. 9, 2015, 1550068, 14 pp.

[6] *F. Pérez-González, J. Rättyä and D. Vukotić*, On composition operators acting between Hardy and weighted Bergman spaces, Expo. Math., **vol. 25**, no. 4, 2007, pp. 309–323.

[7] *J. H. Shapiro*, The essential norm of a compositin operator, Ann. Math., **vol. 127**, no. 2, 1987, pp. 375–404.