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Majorization and Coefficient Problems for a General Class of Starlike Functions

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Abstract: In the current paper, we study a majorization issue for a general category $\mathcal{S}^*(\vartheta)$ of starlike functions, the region of which is often symmetric with respect to the real axis. For various special symmetric functions ϑ, corresponding consequences of the main result are also presented with some relevant connections of the outcomes rendered here with those obtained in recent research. Moreover, coefficient bounds for some majorized functions are estimated.

Keywords: subordination; majorization; starlike function

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1. Introduction and Preliminaries

Let $\mathbb{U}$ denote the unit disk $\{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{H}$ represent the class of analytic functions in $\mathbb{U}$. We denote by $\mathcal{A}$ the subclass of $\mathcal{H}$ consisting of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

Let Φ represent the category of all analytic functions ϖ in $\mathbb{U}$ that satisfy the requirements of $\varpi(0) = 0$ and $|\varpi(z)| < 1$ for $z \in \mathbb{U}$, i.e., we consider Φ the set of Schwarz functions.

Definition 1. *[1,2] For two analytic functions θ and Θ in the unit disk, we state $\theta(z)$ is quasi-subordinate to $\Theta(z)$ if there is a function $\nu(z)$, analytic in $\mathbb{U}$, so that $\theta(z)/\nu(z)$ is analytic in $\mathbb{U}$*

$$\frac{\theta(z)}{\nu(z)} \prec \Theta(z) \quad (z \in \mathbb{U})$$

and $|\nu(z)| \leq 1$ $(z \in \mathbb{U})$, where $\prec$ stands for the usual subordination for analytic functions in $\mathbb{U}$. We denote the above quasi-subordination by

$$\theta(z) \prec_q \Theta(z) \quad (z \in \mathbb{U}). \tag{2}$$

It is remarkable that the relation (2) can be rewritten as follows

$$\theta(z) = \nu(z)\Theta(\varpi(z)) \quad (z \in \mathbb{U}),$$

where $|\nu(z)| \leq 1$ $(z \in \mathbb{U})$ and $\varpi \in \Phi$. For $\nu(z) \equiv 1$ and $\varpi(z) = z$, the quasi-subordination reduces the subordination [3] and the majorization [4], i.e.,

$$\theta(z) = \Theta(\varpi(z)) \quad (z \in \mathbb{U}),$$

written as $\theta(z) \prec \Theta(z)$ and

$$\theta(z) = \nu(z)\Theta(z) \quad (z \in \mathbb{U}),$$

written as $\theta(z) \ll \Theta(z)$, respectively.

Using the principle of subordination, a different subclass $\mathcal{S}^*(\vartheta)$ of starlike functions was defined by Ma and Minda [5] where ϑ is analytic and univalent with $\mathrm{Re}(\vartheta(z)) > 0$ in $\mathbb{U}$, starlike with $\vartheta(0) = 1$ and $\vartheta(\mathbb{U})$ is symmetric with respect to the real axis so that $\vartheta'(0) > 0$. They introduced the class by:

$$\mathcal{S}^*(\vartheta) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \vartheta(z),\ z \in \mathbb{U} \right\}.$$

For example, for the function $\vartheta(z) = (1 + Cz)/(1 + Dz)$ $(-1 \leq D < C \leq 1)$, the class $\mathcal{S}^*(\vartheta)$ becomes the subclass $\mathcal{S}^*[C, D]$ of the well-known Janowski starlike functions. By replacing $C = 1 - 2\gamma$ and $D = -1$ where $0 \leq \gamma < 1$, we obtain the category $\mathcal{S}^*(\gamma)$ of the starlike functions of order γ. Specifically, $\mathcal{S}^* := \mathcal{S}^*(0)$ is the well-known category of starlike functions in $\mathbb{U}$. Some special subclasses of the class $\mathcal{S}^*(\vartheta)$ play a significant act in geometric function theory because of their geometric properties. It is fairly common that a function in one of these subclasses is lying in a given region in the right half-plan and the region is often symmetric with respect to the real axis.

Taking $\vartheta(z) = \sqrt{1+z}$ we get a category of $\mathcal{S}^*_L$, which was reviewed by Sokół and Stankiewicz [6] and implies that $f \in \mathcal{S}^*_L$ if and only if $zf'(z)/f(z) \in \mathrm{B}$, where $\mathrm{B} = \{w \in \mathbb{C} : |w^2 - 1| < 1\}$. Moreover, the features of the category $\mathcal{S}^*_e := \mathcal{S}^*(e^z)$ comprising functions $f \in \mathcal{A}$, with the requirement of $|\log(zf'(z)/f(z))| < 1$ was considered by Mendiratta et al. in [7]. In [8] researchers investigated the category $\mathcal{S}^*(h)$, where

$$h(z) = z + \sqrt{1+z^2} = 1 + z + \frac{z^2}{2} + \cdots,$$

and proved that $f \in \mathcal{S}^*(h)$ if and only if $zf'(z)/f(z) \in \mathrm{R}$, where $\mathrm{R} = \{w \in \mathbb{C} : |w^2 - 1| < 2|w|\}$. Lately, Kanas et al. [9] defined the class $\mathcal{ST}_{hpl}(b) := \mathcal{S}^*(\mathfrak{q}_b(z))$ and obtained some geometric properties in this class where the function

$$\mathfrak{q}_b(z) = \frac{1}{(1-z)^b} = \mathrm{e}^{b\log(1-z)} = 1 + bz + \frac{b(b+1)}{2}z^2 + \frac{b(b+1)(b+2)}{6}z^3 + \cdots \quad (0 < b \leq 1),$$

where the branch of the logarithm is considered by $\mathfrak{q}_b(0) = 1$, maps $\mathbb{U}$ onto a region, which is bounded by a right branch of a hyperbola

$$\mathbb{H}(b) = \left\{ \sigma \mathrm{e}^{\mathrm{i}\chi} : \sigma = \frac{1}{(2\cos(\chi/b))^b},\ |\chi| < \frac{\pi b}{2} \right\}.$$

Moreover, $\mathfrak{q}_b(\mathbb{U})$ is symmetric about the real axis, starlike with respect to $\mathfrak{q}_b(0) = 1$ and convex. Further $\mathfrak{q}_b(z)$ has positive real part in $\mathbb{U}$ and $\mathfrak{q}'_b(0) > 0$. Therefore, $\mathfrak{q}_b(z)$ satisfies the classification of Ma-Minda functions.

Recently, Goel and Kumar [10] introduced the class $\mathcal{S}^*_{SJ}$ and obtained some different problems in this class as follows:

$$\mathcal{S}^*_{SJ} =: \mathcal{S}^*(J) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \frac{2}{1 + \mathrm{e}^{-z}},\ z \in \mathbb{U} \right\}.$$

The modified sigmoid function

$$J(z) = \frac{2}{1+e^{-z}} = 1 + \frac{1}{2}z - \frac{1}{24}z^3 + \cdots,$$

maps $\mathbb{U}$ onto a domain $\Delta_{SJ} := \{\xi \in \mathbb{C} : |\log(\xi/(2-\xi))| < 1\}$, which is symmetric about the real axis. Also, $J(z)$ is a convex function and so starlike function with respect to $J(0) = 1$. Moreover, $J(z)$ has positive real part in $\mathbb{U}$ and $J'(0) > 0$. Therefore, $J(z)$ satisfies the classification of Ma-Minda functions.

MacGregor [4] and Altintas et al. [11] (see also [12]) studied the majorization issues for the category $\mathcal{S}^*$ and for specific analytic functions by convex and starlike functions of complex order.

Theorem 1. ([4], Theorem 1. A) *Let $\theta(z)$ and $\Theta(z)$ be analytic functions in $\mathbb{U}$ with $\theta(z) \ll \Theta(z)$ and $\Theta(0) = 0$. If $0 \le r \le \sqrt{2} - 1$, then*

$$\max_{|z|=r} |\theta'(z)| \le \max_{|z|=r} |\Theta'(z)|.$$

By setting $\Theta(z) = z$, in above outcome we conclude the next well-known result:

Lemma 1. [13] *If $\theta(z)$ be analytic in $\mathbb{U}$ with $|\theta(z)| \le 1$ and $\theta(0) = 0$, then $|\theta'(z)| \le 1$ for $|z| \le \sqrt{2} - 1$.*

Recently, several authors have investigated majorization issues for the families of meromorphic and multivalent meromorphic or univalent and multivalent functions including various linear and nonlinear operators, which all are subordinated by the similar function $\vartheta(z) = (1 + Cz)/(1 + Dz)$ (for example, see [14–20]). Lately, Tang et al. [21] studied majorization problem for the subclasses of $\mathcal{S}^*(\vartheta)$, which are relevant to $\mathcal{S}^*(1 + \sin z)$ and $\mathcal{S}^*(\cos z)$, regardless of any linear or nonlinear operators. Hence, in this work, we study a majorization issue for the general category $\mathcal{S}^*(\vartheta)$ with various special consequences of the main result. Also, some suitable relations of the outcomes are presented with those reported in the earlier results. Moreover, coefficient estimates for majorized functions related to the class $\mathcal{S}^*(\vartheta)$ are obtained.

2. Main Results

We first state and establish a majorization feature for the general category $\mathcal{S}^*(\vartheta)$ and then some consequences of the main result are stated.

Theorem 2. *Let $\theta \in \mathcal{A}$, $\Theta \in \mathcal{S}^*(\vartheta)$ with $\theta(z) \ll \Theta(z)$, then $|\theta'(z)| \le |\Theta'(z)|$ for all z in the disk $|z| \le r_1$, where r_1 is the smallest positive root of the equation*

$$\min_{|z|=r} |\vartheta(z)| (1 - r^2) - 2r = 0, \qquad r \in (0, 1).$$

Proof. Since $\theta(z) \ll \Theta(z)$, considering the concept of majorization, there is a function ν that is analytic in $\mathbb{U}$ with $|\nu(z)| \le 1$ satisfying

$$\theta(z) = \nu(z)\Theta(z).$$

Differentiating the last equality with respect to z, it follows that

$$\theta'(z) = \nu'(z)\Theta(z) + \nu(z)\Theta'(z) = \Theta'(z)\left(\nu'(z)\frac{\Theta(z)}{\Theta'(z)} + \nu(z)\right). \tag{3}$$

Now, let $\Theta \in \mathcal{S}^*(\vartheta)$, then from the subordination concept, there exists a $\varpi \in \Phi$ with $|\varpi(z)| \leq |z| = r$ so that

$$\frac{z\Theta'(z)}{\Theta(z)} = \vartheta(\varpi(z)),$$

or equivalently

$$\frac{\Theta(z)}{\Theta'(z)} = \frac{z}{\vartheta(\varpi(z))}. \tag{4}$$

Since $\mathrm{Re}(\vartheta(z)) > 0$ in $\mathbb{U}$, so $\vartheta(z) \neq 0$ for all $z \in \mathbb{U}$. Now, by the minimum modulus principle we conclude

$$\min_{|z|=r} |\vartheta(z)| \leq \min_{|\varpi(z)|=r} |\vartheta((\varpi(z))| = \min_{|\varpi(z)|\leq r} |\vartheta((\varpi(z))| .$$

We know that ϑ is a continuous function with $\mathrm{Re}(\vartheta(z)) > 0$ in $\mathbb{U}$ and so $\min_{|z|=r} |\vartheta(z)| \neq 0$. Therefore, from this point, (4) and the above relation we obtain

$$\left|\frac{\Theta(z)}{\Theta'(z)}\right| = \frac{|z|}{|\vartheta(\varpi(z))|} \leq \frac{r}{\min_{|z|=r} |\vartheta(z)|}. \tag{5}$$

On the other hand, applying the popular inequality for Schwarz functions, which states that

$$\left|\nu'(z)\right| (1 - |z|^2) \leq 1 - |\nu(z)|^2 . \tag{6}$$

Utilizing (5) and (6) in (3), we obtain

$$\left|\theta'(z)\right| \leq \left(\frac{1 - |\nu(z)|^2}{1 - |z|^2} \frac{r}{\min_{|z|=r} |\vartheta(z)|} + |\nu(z)|\right) \left|\Theta'(z)\right| \qquad (|z| = r < 1).$$

Setting $|\nu(z)| = \gamma$ $(0 \leq \gamma \leq 1)$, it follows that

$$\left|\theta'(z)\right| \leq \left(\frac{1 - \gamma^2}{1 - r^2} \frac{r}{\min_{|z|=r} |\vartheta(z)|} + \gamma\right) \left|\Theta'(z)\right| \qquad (0 \leq \gamma \leq 1).$$

Define

$$l(r, \gamma) = \gamma + \frac{1 - \gamma^2}{1 - r^2} \frac{r}{\min_{|z|=r} |\vartheta(z)|} \qquad (0 \leq \gamma \leq 1,\ 0 < r < 1).$$

In order to determine r_1, we must choose

$$r_1 = \max \{r \in [0, 1) : l(r, \gamma) \leq 1,\ \gamma \in [0, 1]\} .$$

We know $l(r, \gamma) \leq 1$ if and only if

$$0 \leq \min_{|z|=r} |\vartheta(z)| (1 - r^2) - (1 + \gamma)r =: p(r, \gamma).$$

Clearly, the function $p(r, \gamma)$ chooses its minimum value for $\gamma = 1$, that is,

$$\min \{p(r, \gamma) :\ \gamma \in [0, 1]\} = p(r, 1) =: p(r),$$

where

$$p(r) = \min_{|z|=r} |\vartheta(z)| (1 - r^2) - 2r \qquad (0 < r < 1).$$

Further, since $p(0) = 1 > 0$ and $p(1) = -2 < 0$, there exists r_1, so that for all $r \in [0, r_1]$, we have $p(r) \geq 0$ where r_1 is the smallest positive root of the above equality and this completes the proof. □

Remark 1. *Since ϑ is a convex and symmetric with $\mathrm{Re}(\vartheta(z)) > 0$, we get $\min_{|z|=r} |\vartheta(z)| = \vartheta(-r)$ (see [22], Proposition 5.3).*

The following corollary concludes a majorization property for the subclass $\mathcal{ST}_{hpl}(b)$ considering Lemma 2.1 in [9].

Corollary 1. *Let $\theta \in \mathcal{A}$, $\Theta \in \mathcal{ST}_{hpl}(b)$ with $\theta(z) \ll \Theta(z)$. Then for all z in the disk $|z| \leq r_2$, we get $|\theta'(z)| \leq |\Theta'(z)|$, where r_2 is the smallest positive root of the equation*

$$1 - r^2 - 2r(1+r)^b = 0, \qquad r \in (0,1).$$

Example 1. *If we choose the functions*

$$\Theta(z) = z\exp(B_1 z) \in \mathcal{ST}_{hpl}(b) \quad \text{for} \quad 0 < B_1 < 1 - 2^{-b}$$

(see [9]) and

$$\theta(z) = \frac{z}{3+z}\exp(B_1 z),$$

then these functions satisfy in the relation $\theta(z) \ll \Theta(z)$ with $\nu(z) = \frac{1}{3+z}$. Therefore, from Corollary 1 we have

$$\left| \frac{3}{(3+z)^2} + \frac{B_1 z}{3+z} \right| \leq |1 + B_1 z|$$

for $|z| \leq r_2$.

Since $2/(1+e^r) \leq 2/|1+e^{-z}|$ $(|z| = r < 1)$, the next corollary concludes a majorization feature for the subclass $\mathcal{S}^*_{SJ}$.

Corollary 2. *Let $\theta \in \mathcal{A}$, $\Theta \in \mathcal{S}^*_{SJ}$ with $\theta(z) \ll \Theta(z)$. Then $|\theta'(z)| \leq |\Theta'(z)|$ for $|z| \leq r_3$, where r_3 is the smallest positive root of the equation*

$$\frac{2}{1+e^r}(1-r^2) - 2r = 0, \qquad r \in (0,1).$$

Since

$$|\sin z| \leq \sinh r \quad (|z| = r < 1)$$

(see [23]), we have

$$0 < 1 - \sinh r \leq 1 - |\sin z| \leq |1 + \sin z| \quad (|z| = r < 0.8813735870),$$

so the following corollary concludes a majorization property for the subclass $\mathcal{S}^*_s := \mathcal{S}^*(1 + \sin z)$ studied by Cho et al. in [23] and also we have the result which was given by Tang et al. in ([20], Theorem 2.1).

Corollary 3. *Let $\theta \in \mathcal{A}$, $\Theta \in \mathcal{S}^*_s$ with $\theta(z) \ll \Theta(z)$. Then for $|z| \leq r_4$, we get $|\theta'(z)| \leq |\Theta'(z)|$, where r_4 is the smallest positive root of the equation*

$$(1-r^2)(1-\sinh r) - 2r = 0, \qquad r \in (0,1).$$

Example 2. *If we consider the functions*

$$\Theta(z) = z\mathrm{e}^{z/2} \in \mathcal{S}_s^*$$

(see [23]) and

$$\theta(z) = \frac{z}{2+z}\mathrm{e}^{z/2},$$

then we have $\theta(z) \ll \Theta(z)$ with $\nu(z) = \frac{1}{2+z}$. Therefore, from Corollary 3 we get

$$\left|\frac{2}{(2+z)^2} + \frac{z}{2(2+z)}\right| \le \left|1 + \frac{z}{2}\right|,$$

for $|z| \le r_4$.

Since

$$\cos r \le |\cos z| \quad (|z| = r < 1),$$

the following corollary concludes a majorization property for a subclass $\mathcal{S}^*(\cos z)$ and also we have a correction of the result which was given by Tang et al. in ([21], Theorem 2.2).

Corollary 4. *Let $\theta \in \mathcal{A}$, $\Theta \in \mathcal{S}^*(\cos z)$ with $\theta(z) \ll \Theta(z)$. Then $|\theta'(z)| \le |\Theta'(z)|$ for $|z| \le r_5$, where r_5 is the smallest positive root of the equation*

$$(1 - r^2)\cos r - 2r = 0, \qquad r \in (0, 1).$$

In the following corollaries, we obtain majorization properties for two subclasses $\mathcal{S}^*_{\alpha,\mathrm{e}} = \mathcal{S}^*(\alpha + (1-\alpha)\mathrm{e}^z)$ $(0 \le \alpha < 1)$ and $\mathcal{SL}^*(\alpha) = \mathcal{S}^*\left(\alpha + (1-\alpha)\sqrt{1+z}\right)$ $(0 \le \alpha < 1)$, which were defined by Khatter et al. considering Lemma 2.1 in [24]. For $\alpha = 0$, these results reduce to the subclasses $\mathcal{S}^*(\mathrm{e}^z)$ and $\mathcal{S}^*(\sqrt{1+z})$ (see [6,7]).

Corollary 5. *Let $\theta \in \mathcal{A}$, $\Theta \in \mathcal{S}^*_{\alpha,\mathrm{e}}$ with $\theta(z) \ll \Theta(z)$. Then $|\theta'(z)| \le |\Theta'(z)|$ for $|z| \le r_6$, where r_6 is the smallest positive root of the equation*

$$[\alpha + (1-\alpha)\mathrm{e}^{-r}](1 - r^2) - 2r = 0, \qquad r \in (0, 1).$$

Corollary 6. *Let $\theta \in \mathcal{A}$, $\Theta \in \mathcal{SL}^*(\alpha)$ with $\theta(z) \ll \Theta(z)$. Then for $|z| \le r_7$, we get $|\theta'(z)| \le |\Theta'(z)|$, where r_7 is the smallest positive root of the equation*

$$[\alpha + (1-\alpha)\sqrt{1-r}](1 - r^2) - 2r = 0, \qquad r \in (0, 1).$$

The following result concludes a majorization property for a subset $\mathcal{S}^*_{RL} = \mathcal{S}^*(\varphi_0)$ introduced by Mendiratta et al. considering Theorem 2.2 in [25], in which

$$\varphi_0(z) = \sqrt{2} - j\sqrt{\frac{1-z}{1+2jz}} \qquad j = \sqrt{2} - 1,$$

where function φ_0 is a univalent and convex in $\mathbb{U}$.

Corollary 7. *Let $\theta \in \mathcal{A}$, $\Theta \in \mathcal{S}^*_{RL}$ with $\theta(z) \ll \Theta(z)$. Then $|\theta'(z)| \le |\Theta'(z)|$ for $|z| \le r_8$, where r_8 is the smallest positive root of the equation*

$$\varphi_0(r)(1 - r^2) - 2r = 0, \qquad r \in (0, 1).$$

In the following result, we get a majorization property for a category $\mathcal{S}^*(p_l(z))$ introduced by Kanas and Wiśniowska in [26] in which

$$p_l(z) = 1 + P_1(l)z + P_2(l)z^2 + \cdots,$$

where $p_k(z)$ satisfies the conclusion of Remark 1 (see also [27,28]).

Corollary 8. *Let $\theta \in \mathcal{A}$, $\Theta \in \mathcal{S}^*(p_k(z))$ with $\theta(z) \ll \Theta(z)$. Then for $|z| \leq r_9$, we have $|\theta'(z)| \leq |\Theta'(z)|$, where r_9 is the smallest positive root of the equation*

$$p_l(-r)(1-r^2) - 2r = 0, \qquad r \in (0,1).$$

Since $\vartheta(z) = (1+Cz)/(1+Dz)$ satisfies in Remark 1 we obtain a majorization property for the class $\mathcal{S}^*[C,D]$ as follows:

Corollary 9. *Let $\theta \in \mathcal{A}$, $\Theta \in \mathcal{S}^*[C,D]$ with $\theta(z) \ll \Theta(z)$. Then for $|z| \leq r_{10}$, we get $|\theta'(z)| \leq |\Theta'(z)|$, where r_{10} is the smallest positive root of the equation*

$$(1-C)(1-r^2) - 2r(1-D) = 0, \qquad r \in (0,1).$$

To prove the following result, we state the next lemma due to Kuroki and Owa [29] (see also [30]).

Lemma 2. *Let ϑ be a convex in $\mathbb{U}$ with form $\vartheta(z) = 1 + \sum_{n=1}^{\infty} B_n z^n$. If $f \in \mathcal{S}^*(\vartheta)$, then*

$$|a_n| \leq \frac{\prod_{m=2}^{n}(m-2+|B_1|)}{(n-1)!} \qquad (n = 2,3,\cdots).$$

Theorem 3. *Let ϑ be convex in $\mathbb{U}$ and $\theta(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$, $\Theta(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{S}^*(\vartheta)$ with $\theta(z) \ll \Theta(z)$. Then*

$$|a_n| \leq 1 + \sum_{j=2}^{n}\left(\frac{\prod_{m=2}^{j}(m-2+|B_1|)}{(j-1)!}\right) \qquad (n = 2,3,\cdots).$$

Proof. Since $\theta(z) \ll \Theta(z)$, by the majorization principle there is an analytic function $\nu(z) = \sum_{n=0}^{\infty} c_n z^n$ with $|\nu(z)| \leq 1$ satisfying

$$\theta(z) = \nu(z)\Theta(z),$$

where it concludes,

$$a_n = c_0 b_n + c_1 b_{n-1} + \cdots + c_{n-2} b_2 + c_{n-1}. \tag{7}$$

If γ is any circle $|z| = r$, $0 < r < 1$, where $z = re^{i\zeta}$, $0 \leq \zeta \leq 2\pi$, then

$$c_k = \frac{1}{2\pi \mathrm{i}} \int_{\gamma} \frac{\nu(z)}{z^{k+1}} \mathrm{d}z \quad \text{for} \quad k = 0,1,\cdots,n-1.$$

In view of the above equality, we can write the equality (7) in the form (see [4], p. 99)

$$a_n = \frac{1}{2\pi \mathrm{i}} \int_{\gamma} \frac{\nu(z)}{z^n}[1 + b_2 z + \cdots + b_n z^{n-1}]\mathrm{d}z.$$

From the above equality for $n \geq 2$, we obtain

$$|a_n| \leq \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{r^{n-1}}\left|1 + b_2 r e^{i\zeta} + \cdots + b_n r^{n-1} e^{i(n-1)\zeta}\right| \mathrm{d}\zeta$$
$$\leq \frac{1}{r^{n-1}}(1 + |b_2| + \cdots + |b_n|).$$

Since this inequality holds for all r in the interval $0 < r < 1$, it follows that

$$|a_n| \leq 1 + |b_2| + \cdots + |b_n|.$$

Now using Lemma 2 we have

$$|a_n| \leq 1 + \sum_{j=2}^{n}\left(\frac{\prod_{m=2}^{j}(m - 2 + |B_1|)}{(j-1)!}\right),$$

which completes the proof. □

Corollary 10. *Let* $\theta(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$, $\Theta(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{ST}_{hpl}(b)$ *with* $\theta(z) \ll \Theta(z)$. *Then*

$$|a_n| \leq 1 + \sum_{j=2}^{n}\left(\frac{\prod_{m=2}^{j}(m + b - 2)}{(j-1)!}\right) \qquad (n = 2, 3, \cdots).$$

Corollary 11. *Let* $\theta(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$, $\Theta(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{S}^*(e^z)$ *with* $\theta(z) \ll \Theta(z)$. *Then*

$$|a_n| \leq 1 + \sum_{j=2}^{n}\left(\frac{\prod_{m=2}^{j}(m - 1)}{(j-1)!}\right) = n \qquad (n = 2, 3, \cdots).$$

Since the identity function $\Theta(z) = z$ belongs to the category $\mathcal{S}^*(e^z)$, from Corollary 11 we get the next result:

Example 3. *Let* $\theta \in \mathcal{A}$ *and* $|\theta(z)| < 1$, *then*

$$|a_n| \leq n \qquad (n = 2, 3, \cdots).$$

3. Conclusions

In the current paper, we obtain a majorization result for a general category $\mathcal{S}^*(\vartheta)$ of starlike functions. Also, we investigate coefficient bounds for majorized functions associated with the class $\mathcal{S}^*(\vartheta)$. Furthermore, we can consider some particular functions ϑ in Theorems 2 and 3 to get the corresponding majorization results.

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