

Neighborhood properties for k -uniformly starlike functions

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Abstract. In this note, we define the class $S_p(\alpha, k)$ and introduce and investigate coefficient estimates, neighborhood property for functions in the class $S_p(\alpha, k)$. In addition we provide conditions such that the confluent hypergeometric function, belongs to $S_p(\alpha, k)$.

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1. INTRODUCTION

We show the set of all holomorphic functions g in the unit disk $E = \{z : |z| < 1\}$ which are

$$g(z) = z + \sum_{s=2}^{\infty} a_s z^s, \quad (1.1)$$

with \mathcal{A} and let S be the subclass of \mathcal{A} consisting of univalent functions. Suppose that T be the subclass of S which are in the form

$$g(z) = z - \sum_{s=2}^{\infty} a_s z^s \quad (1.2)$$

satisfies the conditions $a_s \geq 0$ ($s = 2, 3, \dots$) with $\sum_{s=2}^{\infty} a_s < 1$.

Also suppose that $S^*(\alpha)$ be the famous subclass of S which are starlike of order α . Indeed $h \in S^*(\alpha)$ is equivalent to $Re(zh'(z)/h(z)) > \alpha$ in E . This subclass has so long history in geometric function theory (for example see [3, 4, 6, 9]).

Let $a, b, c \in \mathbb{C}$ (the set of all complex numbers), such that $c \neq 0, -1, -2, \dots$. It is well known that the answer of the ordinary equation

$$(1 - z)z\varphi''(z) + [c - z(a + b + 1)]\varphi'(z) - ab\varphi(z) = 0$$

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is

$$F(a, b, c; z) = \sum_{s=0}^{\infty} \frac{(a)_s (b)_s}{(c)_s (1)_s} z^s$$

and the function $g(z) = zF(a, b, c; z)$, $z \in E$, is called hypergeometric function. We note that $(a)_0 = 1$ for $a \neq 0$ and $(a)_s = a(a+1)(a+2)\dots(a+s-1)$.

The hypergeometric function plays an important role in various fields. We refer to [8, 10] and references therein for more details about this function.

Finally, for $-1 \leq \alpha \leq 1$ and $k \geq 0$, we introduce a subclass $S_p(\alpha, k)$ of starlike functions in the following way

$$S_p(\alpha, k) = \{g \in S : \operatorname{Re} \left(\frac{zg'(z)}{g(z)} \right) \geq k \left| \frac{zg'(z)}{g(z)} - 1 \right| + \alpha, z \in E\}. \quad (1.3)$$

This class is very famous and important in univalent function theory and relevant subclasses of it have been obtained by many authors such as ([7, 11]). We note that the case $k = 0$ reduce to starlike functions of order α and the case $k = 1$ reduce to uniformly starlike functions of order α . We also let

$$TS_p(\alpha, k) = T \cap S_p(\alpha, k) \quad \text{and} \quad TS^*(\alpha) = T \cap S^*(\alpha).$$

Lemma 1.1. *Let $0 \leq \alpha < 1, k \geq 0$ and $\beta \in \mathbb{R}$. Then $\operatorname{Re}(w) > k|w - t| + \alpha$ is equivalent to $\operatorname{Re}[w(1 + ke^{i\beta}) - kte^{i\beta}] > \alpha$ where w and t are arbitrary complex numbers.*

Lemma 1.2. *Let $k \geq 0$ and $t \in \mathbb{C}$. Then $\operatorname{Re}(t) > k$ is equivalent to $|t - (1 + k)| < |t + (1 - k)|$.*

2. COEFFICIENT BOUNDS

In this section we introduce an inequality that provide a necessary and sufficient Coefficient for functions in the class $TS_p(\alpha, k)$.

Theorem 2.1. *Let $-1 \leq \alpha \leq 1, k \geq 0$ and $g \in TS_p(\alpha, k)$ be in the form (1.2). Then we have*

$$\sum_{s=2}^{\infty} (s(1 - k) + k - \alpha) a_s \leq 1 - \alpha. \quad (2.4)$$

Proof. Let $g \in TS_p(\alpha, k)$ be in the form (1.2). By putting $w = \frac{zg'(z)}{g(z)}$ in (1.3) and by lemma 1.1, we obtain $\operatorname{Re}(w(1 + ke^{i\beta}) - ke^{i\beta}) \geq \alpha$ or

$$\operatorname{Re} \left(\frac{(1 + ke^{i\beta})(1 - \sum_{s=2}^{\infty} s a_s z^{s-1}) - (ke^{i\beta} + \alpha)(1 - \sum_{s=2}^{\infty} a_s z^{s-1})}{1 - \sum_{s=2}^{\infty} a_s z^{s-1}} \right) \geq 0$$

If $z \in E$ is real and tends to 1^- through reals, then we have

$$\operatorname{Re} \left(1 - \alpha + \sum_{s=2}^{\infty} (\alpha - s) a_s + ke^{i\beta} \sum_{s=2}^{\infty} (1 - s) a_s \right) \geq 0.$$

Therefore

$$1 - \alpha - \sum_{s=2}^{\infty} (s - \alpha) a_s + k \sum_{s=2}^{\infty} (s - 1) a_s \geq 0.$$

□

Theorem 2.2. Let $k \geq 0$ and $g \in T$ be an analytic function of the form (1. 2). Then the following condition is sufficient for g to be in the class $S_p(\alpha, k)$.

$$\sum_{s=2}^{\infty} (k(s-1) + s - \alpha) |a_s| \leq 1 \quad (2. 5)$$

if $-1 \leq \alpha < 0$ and

$$\sum_{s=2}^{\infty} (k(s-1) + s - \alpha) |a_s| \leq 1 - \alpha \quad (2. 6)$$

if $0 \leq \alpha \leq 1$.

Proof. By lemma 1.1, we note that the condition (1. 3) is equivalent to $Re(w(1 + ke^{i\beta}) - (\alpha + ke^{i\beta})) \geq 0$ where $w = \frac{zg'(z)}{g(z)}$. So by lemma 1.2, it is sufficient to show that $A \geq B$ where

$$\begin{aligned} A &= |1 + w(1 + ke^{i\beta}) - (\alpha + ke^{i\beta})| \\ &= \left| \frac{(z - \sum_{s=2}^{\infty} a_s z^s) + (1 + ke^{i\beta})(z - \sum_{s=2}^{\infty} sa_s z^s) - (ke^{i\beta} + \alpha)(z - \sum_{s=2}^{\infty} a_s z^s)}{z - \sum_{s=2}^{\infty} a_s z^s} \right| \end{aligned}$$

and

$$\begin{aligned} B &= |1 - w(1 + ke^{i\beta}) + \alpha + ke^{i\beta}| \\ &= \left| \frac{z - \sum_{s=2}^{\infty} a_s z^s - (1 + ke^{i\beta})(z - \sum_{s=2}^{\infty} sa_s z^s) + (ke^{i\beta} + \alpha)(z - \sum_{s=2}^{\infty} a_s z^s)}{z - \sum_{s=2}^{\infty} a_s z^s} \right|. \end{aligned}$$

Let $M = 1/|1 - \sum_{s=2}^{\infty} a_s z^{s-1}|$. Therefore

$$A \geq M(|2 - \alpha| - \sum_{s=2}^{\infty} (k(s-1) + |\alpha - (s+1)|) |a_s|) \quad (2. 7)$$

and

$$B \leq M(|\alpha| + \sum_{s=2}^{\infty} (k(s-1) + |s - (1 + \alpha)|) |a_s|). \quad (2. 8)$$

So by the hypothesis, if $-1 \leq \alpha < 0$, then by (2. 7) and (2. 8)

$$A - B \geq 2M(1 - \sum_{s=2}^{\infty} (k(s-1) + s - \alpha) |a_s|).$$

The last expression is non-negative by (2. 5) and so g belongs to the class $S_p(\alpha, k)$. Also if $0 \leq \alpha \leq 1$, then by (2. 7) and (2. 8) we obtain

$$A - B \geq 2M(1 - \alpha - \sum_{s=2}^{\infty} (k(s-1) + s - \alpha) |a_s|).$$

The last expression is non-negative by (2. 6) and so $g \in S_p(\alpha, k)$. \square

The case $k = 0$ in two previous theorems leads to

Corollary 2.3. Let $g(z) = z - \sum_{s=2}^{\infty} a_s z^s \in T$ and $0 \leq \alpha \leq 1$. Then $g \in S^*(\alpha)$ if and only if $\sum_{s=2}^{\infty} (s - \alpha)a_s \leq 1 - \alpha$.

Theorem 2.4. Let $-1 \leq \alpha \leq 1, 0 \leq k < 1$ and let $g_1(z) = z$,

$$g_s(z) = z - \frac{1 - \alpha}{s(1 - k) + k - \alpha} z^s, s \geq 2.$$

If $g \in TS_p(\alpha, k)$ then we have $g(z) = \sum_{s=1}^{\infty} \lambda_s g_s(z)$ where $\lambda_s \geq 0$ and $\sum_{s=1}^{\infty} \lambda_s = 1$.

Proof. Let $g \in TS_p(\alpha, k)$ has the form $z - \sum_{s=2}^{\infty} a_s z^s$. By Theorem 2.1 we obtain

$$\sum_{s=2}^{\infty} \frac{s(1 - k) + k - \alpha}{1 - \alpha} a_s \leq 1$$

and so

$$a_s \leq \frac{1 - \alpha}{s(1 - k) + k - \alpha}, \quad s \geq 2.$$

Therefore we can set $\lambda_s = \frac{s(1-k)+k-\alpha}{1-\alpha} a_s$ for $s = 2, 3, \dots$ and $\lambda_1 = 1 - \sum_{s=2}^{\infty} \lambda_s$. Thus, $0 \leq \lambda_s \leq 1$ for each $s \in \mathbb{N}$ and $\sum_{s=1}^{\infty} \lambda_s = 1$. Also $g(z)$ has the form

$$\begin{aligned} g(z) &= z - \sum_{s=2}^{\infty} a_s z^s = z - \sum_{s=2}^{\infty} \frac{\lambda_s(1 - \alpha)}{s(1 - k) + k - \alpha} z^s \\ &= \lambda_1 z + \sum_{s=2}^{\infty} \lambda_s \left(z - \frac{1 - \alpha}{s(1 - k) + k - \alpha} z^s \right) \\ &= \sum_{s=1}^{\infty} \lambda_s g_s(z). \end{aligned}$$

□

Theorem 2.5. Let $0 \leq \alpha \leq 1$ and $0 \leq k < 1$. Also let $a, b \in \mathbb{C} - \{0\}$ and $c > |a| + |b| + 1$. Then the condition

$$\frac{\Gamma(c - |a| - |b| - 1)\Gamma(c)}{\Gamma(c - |a|)\Gamma(c - |b|)} ((1 + k)|ab| + (1 - \alpha)(c - |a| - |b| - 1)) \leq 2(1 - \alpha) \quad (2.9)$$

is sufficient for the function $zF(a, b, c; z)$ belongs to $S_p(\alpha, k)$.

Proof. Set $zF(a, b, c; z)$. By Theorem 2.2, we need to show that

$$N := \sum_{s=2}^{\infty} [s(1 + k) - (k + \alpha)] \left| \frac{(a)_{s-1}(b)_{s-1}}{(c)_{s-1}(1)_{s-1}} \right| \leq 1 - \alpha.$$

According to $|(a)_s| \leq (|a|)_s$, we observe that

$$\begin{aligned} N &\leq \sum_{s=2}^{\infty} [s(1+k) - (k+\alpha)] \frac{(|a|)_{s-1}(|b|)_{s-1}}{(c)_{s-1}(1)_{s-1}} \\ &= (1+k) \sum_{s=1}^{\infty} \frac{(s+1)(|a|)_s(|b|)_s}{(c)_s(1)_s} - (k+\alpha) \sum_{s=1}^{\infty} \frac{(|a|)_s(|b|)_s}{(c)_s(1)_s} \\ &= (1+k) \sum_{s=1}^{\infty} \frac{(|a|)_s(|b|)_s}{(c)_s(1)_{s-1}} + (1-\alpha) \sum_{s=1}^{\infty} \frac{(|a|)_s(|b|)_s}{(c)_s(1)_s} \\ &= \frac{|ab|}{c} (1+k) \sum_{s=0}^{\infty} \frac{(1+|a|)_s(1+|b|)_s}{(1+c)_s(1)_s} + (1-\alpha) \sum_{s=1}^{\infty} \frac{(|a|)_s(|b|)_s}{(c)_s(1)_s} \\ &= \frac{|ab|}{c} (1+k) F(1+|a|, 1+|b|, 1+c; 1) + (1-\alpha) (F(|a|, |b|, c; 1) - 1) \\ &= \frac{|ab|}{c} (1+k) \frac{\Gamma(c+1)\Gamma(c-|a|-|b|-1)}{\Gamma(c-|a|)\Gamma(c-|b|)} + (1-\alpha) \frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)} \\ &\quad - (1-\alpha). \end{aligned}$$

Therefore according to (2.9), N is less than $1-\alpha$. □

In the following section we investigate neighborhood property of the functions belongs to the class $TS_p(\alpha, k)$. We remark that this property was introduced by Goodman [5] and Ruscheweyh [12]. See also [1], [2] and [13].

3. NEIGHBORHOOD PROPERTY AND APPLICATIONS

For $\eta \geq 0$ and a function f belonging to \mathcal{A} of the form (1.1), we let (η, ρ) -neighborhood of f by

$$\mathcal{N}_\rho^\eta(f) = \{g \in \mathcal{A} : g(z) = z + \sum_{s=2}^{\infty} b_s z^s, \sum_{s=2}^{\infty} s^\eta |a_s - b_s| \leq \rho\}.$$

For $e(z) = z$, the identity function, we obtain

$$\mathcal{N}_\rho^\eta(e) = \{g \in \mathcal{A} : g(z) = z + \sum_{s=2}^{\infty} b_s z^s, \sum_{s=2}^{\infty} s^\eta |b_s| \leq \rho\}.$$

Theorem 3.1. *Let $0 \leq k < \alpha \leq 1$ and $\eta \leq 1$. Then $TS_p(\alpha, k) \subseteq \mathcal{N}_\rho^\eta(e)$ where $\rho = \frac{2(1-\alpha)}{2-(\alpha+k)}$.*

Proof. Let $g(z) = z - \sum_{s=2}^{\infty} a_s z^s$ be in a class $TS_p(\alpha, k)$. By Theorem 2.1 we obtain

$$\sum_{s=2}^{\infty} (s(1-k) + k - \alpha) a_s \leq 1 - \alpha \tag{3.10}$$

and so

$$\sum_{s=2}^{\infty} s^\eta a_s \leq \sum_{s=2}^{\infty} s a_s \leq \frac{1-\alpha}{1-k} + \frac{\alpha-k}{1-k} \sum_{s=2}^{\infty} a_s. \tag{3.11}$$

On the other hand, from (3. 10), we implies that

$$\sum_{s=2}^{\infty} a_s \leq \frac{1 - \alpha}{2(1 - k) + k - \alpha}. \quad (3. 12)$$

Therefore by (3. 11) and (3. 12) we have

$$\sum_{s=2}^{\infty} s^{\eta} a_s \leq \frac{2(1 - \alpha)}{2 - (\alpha + k)}.$$

□

Corollary 3.2. For $0 \leq \alpha \leq 1$ and $\eta \leq 1$, we have $TS^*(\alpha) \subseteq \mathcal{N}_{\rho}^{\eta}(e)$ where $\rho = \frac{2(1-\alpha)}{2-\alpha}$.

The case $\eta = 1$ in Theorem 3.1 leads to

Corollary 3.3. Let $g(z) = z - \sum_{s=2}^{\infty} a_s z^s \in T$ and $0 \leq k < \alpha \leq 1$. If $\frac{g(z)+\epsilon z}{1+\epsilon} \in S_p(\alpha, k)$, in which $\epsilon > 0$, then

$$\sum_{s=2}^{\infty} s a_s \leq \frac{2(1 - \alpha)(1 + \epsilon)}{2 - (k + \alpha)}$$

and equation is established for the following function

$$g(z) = z - \frac{(1 - \alpha)(1 + \epsilon)}{2 - (k + \alpha)} z^2.$$

Theorem 3.4. Let $g \in T$ and $0 \leq k < \alpha \leq 1$. If $\frac{g(z)+\epsilon z}{1+\epsilon} \in S_p(\alpha, k)$, in which $\epsilon > 0$, then $\mathcal{N}_{\beta}(g)$ is a subset of $S_p(\alpha, k)$ where

$$\beta \leq \frac{1 - \alpha}{1 + k} - \frac{2(1 - \alpha)(1 + \epsilon)}{2 - (k + \alpha)}. \quad (3. 13)$$

Proof. Let $g(z) = z - \sum_{s=2}^{\infty} a_s z^s$ and $f(z) = z + \sum_{s=2}^{\infty} b_s z^s \in \mathcal{N}_{\beta}(g)$. This means that we have $\sum_{s=2}^{\infty} s|a_s + b_s| \leq \beta$. So by the hypothesis and Corollary 3.3, we obtain

$$\begin{aligned} \sum_{s=2}^{\infty} s|b_s| &= \sum_{s=2}^{\infty} s|b_s + a_s - a_s| \\ &\leq \beta + \frac{2(1 - \alpha)(1 + \epsilon)}{2 - (k + \alpha)} \\ &\leq \frac{1 - \alpha}{1 + k}. \end{aligned}$$

Therefore we have

$$\sum_{s=2}^{\infty} \frac{s(1 + k) - (k + \alpha)}{1 - \alpha} |b_s| \leq \frac{1 + k}{1 - \alpha} \sum_{s=2}^{\infty} s|b_s| \leq 1$$

and by Theorem 2.2, $f(z) \in S_p(\alpha, k)$. □

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