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MONOPOLES OVER FUZZY TWO-SPHERE BY ONE SEQUENCE OF THE IRREPS OF SU(2)

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The purpose of this paper is to use the idea in *J. Geom. Phys.* **42**, 54 (2002) to compute the topological charges for a (finite) sequence of noncommutative line bundles over the fuzzy sphere. Central to this task is to construct projective modules associated with sequence of the irreducible sub-representations of the tensor product of two different irreps of SU(2). The topological charges corresponding to such fuzzy line bundles are fractional and different from each other. However, in the commutative limit, those tend to Chern numbers of a sequence of the complex line bundles over two-sphere.

Keywords: Noncommutative geometry; monopoles over fuzzy two-sphere; fuzzy geometry; matrix models.

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1. Introduction

The key feature of quantization is to obtain noncommutative spaces from the commutative ones, by sorting a one-to-one correspondence between an algebra of functions and a sequence of noncommutative algebras.[1] Furthermore, the algebra of quantum operators should be reduced to the algebra of continuous functions in the classical limit. Fuzzy sphere was first introduced by Madore in the context of noncommutative geometry,[2] and was later used in various different physical

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applications.[3–8] The two-sphere as a symplectic manifold is a coadjoint orbit of SU(2) and this allows one to use the adjoint action of the angular momentum operators to define the derivations and consequently the Chern characters associated with projective modules over the fuzzy sphere. In Ref. 6, the theory of quantized equivariant vector bundles over compact coadjoint orbits has been used to define the noncommutative line bundles over the fuzzy sphere. Considering N as a given positive integer number, however allowed to go to infinity, and ν as a fixed non-negative integer or half-integer, the authors have used two irreducible sub-representations of the tensor product of N- and ν-representations of SU(2) to make the fuzzy line bundles. The whole sequences of the projective modules and Chern characters of quantum algebras are generated by changing the parameter N. As it is known, the representation space obtained from the tensor product of two different irreps of SU(2) naturally decomposes into a direct sum of a sequence of subspaces with different highest weight vectors. In spite of the fact that any of the subspaces attributes to itself a projection map on the reducible space, projectors corresponding to only two subspaces with highest and lowest spins, have been utilized to construct the projective modules in Ref. 6. Of course, this method recovers all positive and negative integer (or even zero) Chern numbers of the complex line bundles over the two-sphere in the commutative limit.

This paper is organized as follows: in order to compare with the fuzzy case we briefly review the complex line bundles, projective modules and first Chern numbers for the classical sphere in Sec. 2. Then in Sec. 3, to construct the noncommutative line bundles and their corresponding Chern characters, we use all projections from the tensor product of N- and ν-representations to its irreducible sub-representations. This approach led to the number $2\nu + 1$ of whole infinite sequences of the projective modules over fuzzy two-sphere. Finally, the conclusion is devoted to derive some interesting properties on their topological charges.

2. Complex Line Bundles, Projective Modules and First Chern Numbers for the Classical Sphere

The way to construct the complex line bundles over two-sphere is similar to what has been explained in Refs. 6, 9 and 10. Consider the U(1) Hopf principal fibration π of the total space S^3 over S^2:

$$\mathrm{U}(1) \overset{\text{right U(1)-action}}{\hookrightarrow} S^3 \overset{\pi}{\to} S^2 \,. \tag{2.1}$$

Let $\mathcal{B}_{\mathbb{C}} = \mathcal{C}^\infty(S^3, \mathbb{C})$ and $\mathcal{A}_{\mathbb{C}} = \mathcal{C}^\infty(S^2, \mathbb{C})$ denote the commutative algebras of $\mathbb{C}$-valued smooth functions on the total space S^3 and the base space S^2 under pointwise multiplication, respectively. The sections of the trivial line bundles $S^3 \times \mathbb{C}$ and $S^2 \times \mathbb{C}$ are the continuous functions $\mathcal{C}^0(S^3, \mathbb{C})$ and $\mathcal{C}^0(S^2, \mathbb{C})$, respectively. The elements of $\mathcal{B}_{\mathbb{C}}$ are classified into the right modules of the equivariant maps

$$\mathcal{C}^{\infty}_{(k)}(S^3, \mathbb{C}) = \{\varphi : S^3 \to \mathbb{C}\,,\ \varphi(x \cdot w) = w^{-k} \cdot \varphi(x)\,,\ \forall\, x \in S^3,\ \forall\, w \in \mathrm{U}(1)\} \tag{2.2}$$

over the pull-back of the $\mathcal{A}_{\mathbb{C}}$. The integer number k labels irreps of U(1) on $\mathbb{C}$ and for $k = 0$ we have $\mathcal{C}^{\infty}_{(0)}(S^3, \mathbb{C}) \simeq \mathcal{A}_{\mathbb{C}}$. There is an isomorphism between $\mathcal{C}^{\infty}_{(k)}(S^3, \mathbb{C})$ and the smooth sections $\Gamma^{\infty}(S^2, L^k)$ of the associated complex line bundles

$$L^k := S^3 \times_k \mathbb{C} \xrightarrow{\mathbf{p}} S^2 \tag{2.3}$$

over the two-sphere.[11] Also, the Serre–Swan's theorem states that for a compact smooth manifold, there is a complete equivalence between the category of smooth line bundles and the category of finitely generated projective modules.[12] Therefore, for the associated line bundle L^k the left $\mathcal{A}_{\mathbb{C}}$-module of sections $\Gamma^{\infty}(S^2, L^k)$ is equivalent to the image in the free module $(\mathcal{A}_{\mathbb{C}})^n = \mathcal{A}_{\mathbb{C}} \otimes \mathbb{C}^n$ of a self-adjoint idempotent operator p_k: $\Gamma^{\infty}(S^2, L^k) = (\mathcal{A}_{\mathbb{C}})^n p_k$ with $n = |k| + 1$. This means that every element $\psi \in \Gamma^{\infty}(S^2, L^k)$ can be presented as an element of $\psi \in (\mathcal{A}_{\mathbb{C}})^n$ with $\psi = \psi p_k$. The projector $p_k \in \mathbb{M}_n(\mathcal{A}_{\mathbb{C}})$ is invariant under the similarity transformation induced by the U(1) right action and explicitly constructed by using the equivariant maps $\mathcal{C}^{\infty}_{(k)}(S^3, \mathbb{C})$.[10]

Let $(\Omega^*(\mathcal{A}_{\mathbb{C}}), d)$ be a differential calculus on $\mathcal{A}_{\mathbb{C}}$. Then for the Grassmann connection

$$\nabla_k = p_k \circ d : \Gamma^{\infty}(S^2, L^k) \to \Omega^1(\mathcal{A}_{\mathbb{C}}) \otimes_{\mathcal{A}_{\mathbb{C}}} \Gamma^{\infty}(S^2, L^k) \tag{2.4}$$

obeying Leibnitz rule, $\nabla_k(f\psi) = f\nabla_k\psi + df \otimes_{\mathcal{A}_{\mathbb{C}}} \psi p_k$ for all $f \in \mathcal{A}_{\mathbb{C}}$ and $\psi \in \Gamma^{\infty}(S^2, L^k)$, we can associate the curvature operator as an $\mathcal{A}_{\mathbb{C}}$-linear map,[13,14]

$$\nabla_k^2 = p_k\, dp_k \wedge dp_k : \Gamma^{\infty}(S^2, L^k) \to \Omega^2(\mathcal{A}_{\mathbb{C}}) \otimes_{\mathcal{A}_{\mathbb{C}}} \Gamma^{\infty}(S^2, L^k)\,. \tag{2.5}$$

The endomorphisms of the right $\mathcal{A}_{\mathbb{C}}$-module of sections $\Gamma^{\infty}(S^2, L^k)$, i.e. $\mathrm{End}_{\mathcal{A}_{\mathbb{C}}}(\Gamma^{\infty}(S^2, L^k))$, are used as usual to define a two-form of the curvature tensor as

$$\mathbf{F}_k := \mathrm{Tr}(p_k\, dp_k \wedge dp_k) \in \Omega^2(\mathcal{A}_{\mathbb{C}})\,, \tag{2.6}$$

in which Tr denotes trace of the endomorphism. It is immediate that $\mathbf{F}_k$ is a cocycle, and it defines a first cohomology class, not depending on the choice of sections: $[\mathbf{F}_k] \in H^2(\mathcal{A}_{\mathbb{C}})$. The corresponding Chern character of the module $\Gamma^{\infty}(S^2, L^k)$ with respect to $(\Omega^*(\mathcal{A}_{\mathbb{C}}), d)$ is given as $C_1(p_k) = \mathbf{F}_k$ and by integrating over S^2, the first Chern number corresponding to the associated complex line bundle L^k is calculated as below (see Ref. 10 for the explicit calculations)

$$c_1(p_k) = \frac{-1}{2\pi i} \int_{S^2} C_1(p_k) = -k \in \mathbb{Z}\,. \tag{2.7}$$

k distinguishes both the different magnetic charges of the Dirac monopole in $\mathbb{R}^3$ and inequivalent line bundles over the two-dimensional sphere S^2. The complex line bundles L^k over the two-sphere with $k \neq 0$ are non-trivial and projectors corresponding to L^k and L^{-k} are the transpose of each other. It is also reminded that the rank of the module $\Gamma^\infty(S^2, L^k)$ is given by the zero's Chern class: $C_0(p_k) = \mathrm{Tr}(p_k)$.

3. Fuzzy Line Bundles, Chern Characters and Their Topological Charges: Formulation for a Sequence of the Irreps of SU(2)

Along the line of Ref. 6, we extend the construction of noncommutative equivariant complex vector bundles on coadjoint orbits in a manner that allows us to present all possible projectors corresponding to different modules over the matrix algebra of the fuzzy sphere. Denote the unitary irrep space of SU(2) with highest weight N as $[N]$ with $\dim([N]) = 2N + 1$. The property $[N^*] = [N]^*$ follows from the fact that the linear dual of an irrep is also an irrep and we can therefore conclude that $\mathrm{End}([N]) = [N] \otimes [N]^*$. Let $(2N + 1) \times (2N + 1)$-matrices X_1, X_2 and X_3 with the well-known commutation relations be Hermitian generators of su(2) algebra. The algebra $\mathcal{A}_N$ on the fuzzy sphere[1,2] is the noncommutative matrix algebra $\mathrm{Mat}(2N + 1)$ which is generated by all endomorphisms of $[N]$, a complete operator basis of $\mathcal{A}_N := \mathrm{End}([N])$. $\mathcal{A}_N$ as an su(2)-module can be decomposed into the direct sum of irreducible su(2)-modules: $\mathcal{A}_N = [0] \oplus [1] \oplus \cdots \oplus [2N]$. The noncommutative coordinates of the fuzzy sphere are represented by $Y_a = \frac{X_a}{\sqrt{N(N+1)}}$ satisfying the commutation relations as well as the following condition:

$$[Y_a, Y_b] = \frac{i\epsilon_{abc}}{\sqrt{N(N+1)}} Y_c \quad \text{and} \quad Y_a Y_a = 1\,, \qquad a, b, c = 1, 2, 3\,. \tag{3.1}$$

$\mathcal{A}_N$'s form the bundle $\mathcal{A}_{\mathbb{N}}$ over the discrete parameter space $\mathbb{N}$ and this allows us to consider the generators X_a and Y_a as unbounded and bounded sections of the bundle, respectively.[15] Conceptually, this is the same as the previous section, we just need to define a differential calculus on $\mathcal{A}_N$ in order to construct p-forms over that. Denote by $\mathrm{Der}(\mathcal{A}_N)$ the $\mathbb{C}$-vector space of all derivations of $\mathcal{A}_N$. e_a's defined as $e_a(\phi) := \mathrm{ad}_{X_a}\phi$ describe the derivations e_a along X_a of $\phi \in \mathcal{A}_N$ and are the fuzzy analogue of the classical derivations ("vector fields") $\mathcal{L}_a = i\epsilon_{abc}x_b\partial/\partial x_c$ induced by the usual action of SO(3) on two-sphere. The fuzzy derivations e_a form the three-dimensional $\mathbb{C}$-vector subspace $\mathrm{Der}_3(\mathcal{A}_N)$ of $\mathrm{Der}(\mathcal{A}_N)$. The relation $[e_a, e_b] = i\epsilon_{abc}e_c$ follows immediately from the Jacobi identity, that is the Leibnitz rule for the three natural derivations of functions. In $N \to \infty$ limit the tangent space becomes two-dimensional. Denote by $\Omega^p_{(N)}$ the set of all p-forms over $\mathcal{A}_N$. The exterior derivative $d : \mathcal{A}_N \to \Omega^1_{(N)}$ is given by $d\phi = e_a(\phi)\Theta_a$ with Θ_a as the basis one-forms dual to the vector fields e_a, $\Theta_b(e_a) = \delta_{ba}\mathbb{I}$. The exterior derivative is also extended to $\Omega^*_{(N)} := \bigoplus_p \Omega^p_{(N)}$ by linearity and the graded Leibnitz rule. The matrix-valued one-form Θ as the analog of Maurer–Cartan one-form is defined by $\Theta(e_a) = -X_a$ (or $\Theta = -X_a\Theta_a$) and simply satisfies $d\Theta + \Theta \wedge \Theta = 0$. This allows us to introduce the exterior derivative of a zero function $\phi \in \Omega^0_{(N)} = \mathcal{A}_N$ as $d\phi = [\phi, \Theta]$.

Now let $[\nu]$ be a spin-ν irrep of su(2) algebra with $N > \nu$. The sphere is a coadjoint orbit with a nonzero second cohomology class. Therefore, as previously it has been proven in Ref. 15 that we can quantize equivariant vector bundles over coadjoint orbits by means of the orthogonal projectors $p_j \in \mathrm{Hom}_{\mathrm{SU}(2)}([N] \otimes [\nu], [N-\nu+j]) \cong \mathcal{A}_N \otimes \mathrm{End}[\nu]$ with $j = 0, 1, 2, \ldots, 2\nu$. For a given j, precisely one copy of $[N-\nu+j]$ as jth irrep of $[N] \otimes [\nu] = \bigoplus_{k=0}^{2\nu} [N-\nu+k]$ always exists, therefore, the orthogonal projectors $p_j : [N] \otimes [\nu] \to [N-\nu+j]$ can be applied to construct the number $2\nu+1$ of noncommutative line bundles $\mathbf{L}_N^{2(j-\nu)} := (\mathcal{A}_N \otimes [\nu])p_j$ over the fuzzy sphere. This process for a given j, gives a whole sequence of the projective modules over the matrix algebras $\mathcal{A}_N$ of the fuzzy sphere. Furthermore, for a given N, we have a (finite) sequence of noncommutative line bundles: $\mathbf{L}_N^{-2\nu}, \mathbf{L}_N^{-2(\nu-1)}, \ldots, \mathbf{L}_N^{2\nu}$. Generally, $\mathbf{L}_N^{2(j-\nu)}$ is a finitely generated projective left $\mathcal{A}_N$-module and in the commutative limit $N \to \infty$ for a spin number set as $\nu = j - \frac{k}{2}$, it tends to the module of sections of L^k.

Here, we are in a position that for any of the orthogonal projectors $p_0, p_1, \ldots, p_{2\nu}$ we can reconstruct the three lemmas considered in Ref. 6 and follow the proofs along its lines. In the sequel, let $\pi_j : \mathrm{SU}(2) \to \mathrm{End}([N-\nu+j])$ be the jth unitary irrep of SU(2) group with highest weight vector $|h_{N-\nu+j}\rangle$ embedded in $[N] \otimes [\nu]$. It helps us to define the explicit forms of the orthogonal projectors $p_j : [N] \otimes [\nu] \to [N-\nu+j]$ as follows:

$$p_j = (2(N-\nu+j)+1) \int_{g \in \mathrm{SU}(2)} \pi_j(g) |h_{N-\nu+j}\rangle \langle h_{N-\nu+j}| \pi_j^{-1}(g) d\mu(g) \,, \tag{3.2}$$

in which $d\mu$ denotes the normalized Haar measure on SU(2). Also, the first components of the Chern characters associated with projective modules over the fuzzy sphere are defined as

$$\mathbf{F}_j^{N,\nu} = \mathrm{Tr}_2(p_j\, dp_j \wedge dp_j) \in \mathcal{A}_N \otimes ((\mathrm{Der}_3(\mathcal{A}_N))^* \wedge (\mathrm{Der}_3(\mathcal{A}_N))^*) \,, \tag{3.3}$$

where Tr_2 means the trace in $\mathrm{End}([\nu])$ and "d" acts only on the $\mathcal{A}_N$ part of p_j. According to Lemma 2 of Ref. 6, for every unitary irrep $[N-\nu+j]$ there is a specific proportional coefficient f_j depending on N, ν and j so that the Chern character $\mathbf{F}_j^{N,\nu}$ satisfies the following relations:

$$\mathbf{F}_j^{N,\nu} = f_j \epsilon_{abc} X_c \Theta_a \wedge \Theta_b = \frac{(N(N+1))^{\frac{3}{2}}}{\frac{1}{2} - N(N+1)} f_j \epsilon_{abc} Y_a \, dY_b \wedge dY_c \,. \tag{3.4}$$

They imply that the Chern character $\mathbf{F}_j^{N,\nu}$ is SU(2)-equivariant map from $\mathrm{sl}(2,\mathbb{C}) \otimes \mathrm{sl}(2,\mathbb{C})$ to the matrix algebra $\mathcal{A}_N$ and consequently is unique up to a factor f_j. The most immediate result from (3.3) and the first relation of (3.4) is

$$\mathrm{Tr}_2(p_j\, dp_j(e_a) dp_j(e_b)) = f_j \epsilon_{abd} X_d \,, \tag{3.5}$$

which in turn gives an explicit expression for f_j,

$$f_j = \frac{\epsilon_{abc}\, \mathrm{Tr}(p_j\, dp_j(e_a) dp_j(e_b) X_c)}{2N(N+1)(2N+1)} \,, \tag{3.6}$$

where Tr denotes the matrix trace on End([N]) ⊗ End([ν]). Then, from Lemma 3 of Ref. 6 one concludes that

$$f_j = \frac{(2(N-\nu+j)+1)}{2N(N+1)(2N+1)}\epsilon_{abc}\langle h_{N-\nu+j}|[X_a,p_j][X_b,p_j]X_c|h_{N-\nu+j}\rangle$$
$$= \frac{(2(N-\nu+j)+1)}{2N(N+1)(2N+1)}[\epsilon_{abc}\langle h_{N-\nu+j}|X_a p_j X_b p_j X_c|h_{N-\nu+j}\rangle$$
$$- i\langle h_{N-\nu+j}|X_a p_j X_a|h_{N-\nu+j}\rangle]\,. \tag{3.7}$$

Defining the noncommutative volume form as $\omega := \frac{1}{8\pi}\epsilon_{abc}Y_a\,dY_b \wedge dY_c \in \Omega^2(\mathcal{A}_N)$ and the integral by $\int \phi\omega := \frac{\mathrm{Tr}(\phi)}{2N+1}$ for any $\phi \in \mathcal{A}_N$, it becomes obvious that $\int \omega = 1$. Finally, with the help of the second relation of (3.4) we can calculate the topological charges corresponding to the nontrivial fuzzy line bundles $\mathbf{L}^{2(j-\nu)}$ as

$$c_1^{N,\nu}(p_j) = \frac{-1}{2\pi i}\int \mathbf{F}_j^{N,\nu} = 4i\frac{(N(N+1))^{\frac{3}{2}}}{\frac{1}{2}-N(N+1)}f_j\,. \tag{3.8}$$

Now, it is time to calculate explicitly f_j in terms of N, ν and j.

4. Explicit Calculations for the Topological Charges of All Fuzzy Line Bundles

Consider the boson creation and annihilation operators $a_i^\dagger = z_i$ and $a_i = \frac{\partial}{\partial z_i}$ for a two-mode harmonic oscillator with the commutators $[a_i, a_j^\dagger] = \delta_{ij}$, which give a realization of SU(2) algebra as

$$X_+ = a_1^\dagger a_2\,, \qquad X_- = a_2^\dagger a_1\,, \qquad X_3 = \frac{1}{2}(a_1^\dagger a_1 - a_2^\dagger a_2)\,, \tag{4.1}$$

with the following commutation relations

$$[X_+, X_-] = 2X_3\,, \qquad [X_3, X_\pm] = \pm X_\pm\,. \tag{4.2}$$

The Hilbert space $\mathcal{H}_n$ spanned by the homogeneous polynomials of two complex variables z_1 and z_2 of fixed degree $n \in \mathbb{N}$ $(k = 0, 1, 2, \ldots, n)$,

$$|\psi_k^n\rangle := \sqrt{\binom{n}{k}}\, z_1^k z_2^{n-k} \quad \text{with} \quad \langle \psi_k^n|\psi_{k'}^n\rangle = \delta_{kk'}\,, \tag{4.3}$$

realizes the unitary irrep $\left[\frac{n}{2}\right]$ of SU(2) with highest weight $\frac{n}{2}$ (see Ref. 16 for more details on the inner product),

$$X_+|\psi_k^n\rangle = \sqrt{(n-k)(k+1)}|\psi_{k+1}^n\rangle\,,$$
$$X_-|\psi_{k+1}^n\rangle = \sqrt{(n-k)(k+1)}|\psi_k^n\rangle\,, \qquad X_3|\psi_k^n\rangle = \left(k - \frac{n}{2}\right)|\psi_k^n\rangle\,. \tag{4.4}$$

Following the previous section, let us consider the reducible representation $[N] \otimes [\nu]$ on the Hilbert space $\mathcal{H}_{2N} \otimes \mathcal{H}_{2\nu}$ with an explicit realization of the SU(2) generators as $X_{\pm,3} \otimes \mathbb{I} + \mathbb{I} \otimes X_{\pm,3}$. It is first necessary to note that the most general form of the highest vector belonging to $[N-\nu+j]$ can be proposed as

$$|h_{N-\nu+j}\rangle = \sum_{k=0}^{2\nu-j} \mathcal{C}_k |\psi^{2N}_{2N-k}\rangle \otimes |\psi^{2\nu}_{j+k}\rangle \,, \tag{4.5}$$

in which $\mathcal{C}_k$'s are the normal Clebsch–Gordon coefficients of SU(2). The satisfaction of the annihilation relation $(X_+ \otimes \mathbb{I} + \mathbb{I} \otimes X_+)|h_{N-\nu+j}\rangle = 0$ requires a recursion relation on the Clebsch–Gordon coefficients as

$$\mathcal{C}_k = -\sqrt{\frac{(j+k)(2\nu-j-k+1)}{k(2N-k+1)}}\,\mathcal{C}_{k-1} \tag{4.6}$$

whose solution is

$$\mathcal{C}_k = (-1)^k \sqrt{\binom{j+k}{k}\binom{2\nu-j}{k}\binom{2N}{k}^{-1}}\,\mathcal{C}_0 \,. \tag{4.7}$$

The constant $\mathcal{C}_0$ is fixed by the normalization condition $\langle h_{N-\nu+j}|h_{N-\nu+j}\rangle = 1$ as follows:

$$\mathcal{C}_0 = \sqrt{\binom{2N}{2\nu-j}\binom{2N+j+1}{2\nu-j}^{-1}} \,, \tag{4.8}$$

in which we have used the following relation[17]

$$\sum_{k=0}^{a} \binom{a}{k}\binom{b+k}{k}\binom{c}{k}^{-1} = \binom{b+c+1}{a}\binom{c}{a}^{-1} \tag{4.9}$$

with a, b and c as positive integer numbers. Defining

$$\begin{aligned} |v\rangle^j &:= (X_- \otimes \mathbb{I} + \mathbb{I} \otimes X_-)|h_{N-\nu+j}\rangle \,, \\ |w\rangle^j &:= (X_1 \otimes \mathbb{I})|h_{N-\nu+j}\rangle = |w\rangle^j_+ + |w\rangle^j_- \,, \end{aligned} \tag{4.10}$$

with

$$|w\rangle^j_\pm := \frac{1}{2}\sum_{k=0}^{2\nu-j} \mathcal{C}_k \sqrt{\left(2N-k+\frac{1}{2}\pm\frac{1}{2}\right)\left(k+\frac{1}{2}\mp\frac{1}{2}\right)}\,|\psi^{2N}_{2N-k\pm1}\rangle \otimes |\psi^{2\nu}_{j+k}\rangle \,, \tag{4.11}$$

it is straightforward to show that

$$(X_3 \otimes \mathbb{I} + \mathbb{I} \otimes X_3)|w\rangle^j_\pm = (N-\nu+j\pm1)|w\rangle^j_\pm \tag{4.12}$$

and

$$\begin{aligned} {}^j\langle v|v\rangle^j &= 2(N-\nu+j)\,, \\ \lambda_j := \frac{{}^j\langle v|w\rangle^j_-}{{}^j\langle v|v\rangle^j} &= \frac{j^2+(2N-2\nu+1)j+2N^2+2N(1-\nu)-2\nu}{4(N-\nu+j)(N-\nu+j+1)} \,. \end{aligned} \tag{4.13}$$

Note that we have used the following relation[17] to obtain the last relation of (4.13),

$$\sum_{k=0}^{a} \frac{(c-k)(k+1)-(a-k)(b-a+k+1)}{k+1}(c-k)\binom{a}{k}\binom{b-a+k}{k}\binom{c}{k+1}^{-1}$$
$$= [2a^2 - 2(b+c+1)a + (b+c+2)c]\binom{b+c-a+1}{a}\binom{c}{a}^{-1}. \qquad (4.14)$$

Also, if we define

$$|K_j\rangle := (N - X_3 \otimes \mathbb{I})|h_{N-\nu+j}\rangle = \sum_{k=0}^{2\nu-j} k\mathcal{C}_k |\psi^{2N}_{2N-k}\rangle \otimes |\psi^{2\nu}_{j+k}\rangle\,, \qquad (4.15)$$

then it is easy to show that

$$(X_3 \otimes \mathbb{I} + \mathbb{I} \otimes X_3)|K_j\rangle = (N - \nu + j)|K_j\rangle \qquad (4.16)$$

and

$$p_j(X_3 \otimes \mathbb{I})|h_{N-\nu+j}\rangle = (N - \mu_j)|h_{N-\nu+j}\rangle$$

with

$$\mu_j = \langle h_{N-\nu+j}|K_j\rangle = \frac{(j+1)(2\nu-j)}{2(N-\nu+j+1)}\,. \qquad (4.17)$$

Finally, the second relation of (3.7) gives

$$f_j = \frac{2(N-\nu+j)+1}{2N(N+1)(2N+1)}[2i\lambda_j^2[(2N - 2\mu_j - 1)\,{}^j\langle v|v\rangle^j$$
$$- {}^j\langle v|X_3|v\rangle^j] - i(N-\mu_j)^2]\,. \qquad (4.18)$$

Consequently from (3.8) and some of the above results, the topological charges corresponding to the fuzzy line bundles $\mathbf{L}^{2(j-\nu)}$ are calculated as fractional numbers,

$$c_1^{N,\nu}(p_j) = -\frac{[2(N-\nu+j)+1][2(N-\nu)(j-\nu)+j(j+1)][2N^2+2N(j-\nu+1)+(j+1)(j-2\nu)]^2}{2N^{\frac{-1}{2}}(N+1)^{\frac{-1}{2}}(2N+1)(2N^2+2N-1)(N-\nu+j)^2(N-\nu+j+1)^2}\,. \qquad (4.19)$$

(4.19) contains the results obtained in Ref. 6 for the irreducible sub-representations with the highest and lowest spins corresponding to $j = 2\nu$ and $j = 0$.

5. Concluding Remarks

- The sequence of projective left $\mathcal{A}_N$-modules $\mathbf{L}_N^{-2\nu}, \mathbf{L}_N^{-2(\nu-1)}, \ldots, \mathbf{L}_N^{2\nu}$ over the fuzzy sphere in the commutative limit $N \to \infty$ approaches to the sequence of the sections of the associated complex line bundles $L^{-2\nu}, L^{-2(\nu-1)}, \ldots, L^{2\nu}$ over the two-sphere, respectively, since we can see directly from (4.19) that ($j = 0, 1, 2, \ldots, 2\nu$)

$$\lim_{N\to\infty} c_1^{N,\nu}(p_j) = c_1(p_{2(j-\nu)}) = -\lim_{N\to\infty} c_1^{N,\nu}(p_{2\nu-j})\,, \qquad (5.1)$$

in which $c_1(p_{2(j-\nu)}) = 2(\nu - j)$ follows from (2.7). Equation (5.1) implies that the Whitney sum of the two noncommutative line bundles $\mathbf{L}_N^{2(j-\nu)}$ and $\mathbf{L}_N^{2(\nu-j)}$ is transformed to a trivial line bundle in the commutative limit. So in this sense, the appearance of the topological charges over two-sphere at both positive and negative integers is consistent (because of symmetry) with the theory of irreps of SU(2).

- According to (5.1) in the limiting process, the zero topological charge for the classical sphere follows from the νth unitary irrep of SU(2) group with highest weight vector $|h_N\rangle$. So, a trivial bundle over the classical sphere is a limiting case of the fuzzy line bundle constructed by a unique irreducible sub-representation of $[N] \otimes [\nu]$ with ν as a non-negative integer number.
- However, contrary to the limiting case, the signs of $c_1^{N,\nu}(p_j)$ and $c_1^{N,\nu}(p_{2\nu-j})$ are not always different from each other. It is simple to conclude that for $\nu < N < \frac{j(j+1)}{2(\nu-j)} + \nu$ and $\nu < N < \frac{j(j+1)}{2(j-\nu)} - \nu - 1$ with $\nu \neq 1, \frac{1}{2}, 2$, these topological charges have the same negative sign, when $\nu > j$ and $j > \nu$, respectively.
- It is straightforward to conclude that a nonzero (fuzzy) topological charge cannot be obtained by taking the limit $\nu \to 0$ in the expression (4.19).

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