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From the generalized uncertainty relations on fuzzy AdS_2 to the Poincaré geometry

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Abstract The positive and discrete unitary irreps of $SU(1,1)$ are used to construct fuzzy (Euclidean) AdS_2. Two different types of uncertainty relation involving the Weyl–Heisenberg and a weaker type are studied. It is shown that there are no generalized coherent states which simultaneously minimize the Weyl–Heisenberg uncertainty relations among three non-commuting embedding coordinates of the fuzzy AdS_2. However, generalized squeezed states that simultaneously satisfy the three weaker uncertainty relations do exist, and reproduce some properties of the classical AdS_2. Up to a common scaling factor in terms of the irrep label, the expectation values of the non-commuting coordinates over such states are described in the same manner as the classical AdS_2, in terms of the Poincaré coordinates. The expectation values on the fuzzy AdS_2 tend to their corresponding values in the commutative limit.

1 Introduction

Non-commutative spaces in physics were initially introduced to regularize the divergences encountered in field theory by geometric quantization of the manifolds. They possess a rich mathematical structure which can be elegant and highly important for the solutions of various problems. When the effective constant of quantization tends to zero, the fuzzy manifold, as a quantum representation of classical manifold, is transformed to the classical one. Many systems of interest in physics have configuration spaces with the form of simple or semi-simple Lie groups. $SO(3)$ and $SL(2,R)$ are the symmetry groups for two of the most famous surfaces, the so-called the sphere $S^2 = SU(2)/U(1)$ and hyperbolic plane $AdS_2 = SU(1,1)/U(1)$. Many studies have been made on the fuzzy version of the first since it is compact and therefore the fuzzy sphere becomes a finite-dimensional matrix algebra on which the Lie group $SO(3)$ acts in simple ways. The fuzzy sphere was first introduced by Madore in the context of non-commutative geometry [1], and was later used in various different physical applications [2–7]. A different method cooperating spectral triples has been used to study the spectrum of Dirac operator, differential calculus, Chern character and topological charges on the fuzzy sphere [8–12]. Furthermore, there are a few studies that have examined different methods to formulate the non-commutative analog of a noncompact surface with constant negative curvature [13–15]. The fuzzy AdS_2 appears as a descendant of the 4-dimensional extremal black hole with unusual quantum properties. For this model, the quantum analog of the Laplace equation is solved and a strong consistency check is provided by calculating three point vertices and demonstrating the conventional $1/N$ expansion as well as non-perturbative effects in large N [16]. The fuzzy AdS_2 model is also applied to derive holography from space-time non-commutativity [17]. Besides, in Ref. [18], the Landau problem has been formulated on the fuzzy AdS_2 and there it has been shown that the degeneracy of the discrete spectrum in the commutative AdS_2 is lifted in the fuzzy case. The Watamuras' approach along with pseudo-Hermiticity have been used to study the spectrum of Dirac operator on the fuzzy AdS_2 too [19].

The so-called generalized squeezed states, which simultaneously minimize an uncertainty relation weaker than the Weyl–Heisenberg one for all pairs of the three embedding coordinates of the fuzzy sphere S_F^2, have been found in Ref. [20]. They have been parameterized by an appropriate complex stereographic coordinate, described in terms of the highest and second highest bases of $su(2)$ Lie algebra, and give expectation values for the embedding coordinates of S_F^2 in accordance with the stereographic projec-

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tion $S^2 \to C$. These generalized squeezed states have also been used to construct the Grosse–Presnajder associative star product which involves a finite number of derivatives on its primary functional space [21]. The procedure of studying the fuzzy AdS_2 in this paper is based on the approach of [20].

The paper is organized as follows: in Sect. 2, we briefly review the relations among the coordinates in the Poincaré upper half-plane and the Poincaré disc with the embedding coordinates in 3-dimensional flat Minkowskian space of the 2-dimensional hyperbolic geometry. Then in Sect. 3, to construct fuzzy AdS_2 and the space of all analytic functions of coordinates and derivations on that, we use the positive principal j-irreps of $SU(1,1)$. In this approach, the functions are replaced by Hermitian matrices of infinite order on the fuzzy AdS_2. It is shown that j measures the size of the Planck scale and the fuzzy AdS_2 is reduced to the classical manifold for $j \to \infty$. Then in Sect. 4, we give a brief review on two different types of inequality: the Weyl–Heisenberg and its weaker type. Later, for every type in Sect. 4, we will look for the states which simultaneously minimize three possible inequalities among the three non-commuting coordinates of fuzzy AdS_2. If there exist such states, they are called the generalized coherent and generalized squeezed states, respectively. In Sect. 5, we first show that the generalized coherent states do not exist. Then the generalized squeezed states on the fuzzy AdS_2 are calculated in terms of the Poincaré disc coordinates. Finally, Sect. 6 is devoted to calculate the expectation values of the embedding coordinates of fuzzy AdS_2 over the squeezed states. It is shown that the classical embedding coordinates are derived from the fuzzy ones by the limiting process $j \to \infty$.

2 The Poincaré coordinates for the 2-dimensional hyperbolic geometry

The Euclidean $AdS_2 = SU(1,1)/U(1)$ is defined by its embedding in 3-dimensional flat Minkowskian space

$$x_i \eta^{ij} x_j = x_1^2 + x_2^2 - x_3^2 = -1, \quad i, j = 1, 2, 3, \tag{1}$$

in which $\eta^{ij} = \eta_{ij} = \mathrm{diag}(1, 1, -1)$ is the Minkowskian metric which raises and lowers the indices. This is also known as the Poincaré upper half-plane $H^2 = \{z = X + iY \in R^2 : Y > 0\}$ with the Riemannian metric $g_{z\bar{z}} = (\frac{2i}{z-\bar{z}})^2$ and negative scalar curvature $R = -2$ (the bar indicates the complex conjugation):

$$ds^2 = g_{z\bar{z}}\, dz\, d\bar{z} = \frac{dX^2 + dY^2}{Y^2}. \tag{2}$$

The points in Poincaré upper half-plane are mapped to points in the AdS_2 as

$$x_1 = -i\frac{z+\bar{z}}{z-\bar{z}}, \qquad x_2 = i\frac{1-|z|^2}{z-\bar{z}}, \qquad x_3 = i\frac{1+|z|^2}{z-\bar{z}}. \tag{3}$$

Furthermore, if we make the following coordinates transformations:

$$X = \frac{-2\tanh\frac{r}{2}\cos\phi}{1 + 2\tanh\frac{r}{2}\sin\phi + \tanh^2\frac{r}{2}}, \qquad Y = \frac{1-\tanh^2\frac{r}{2}}{1 + 2\tanh\frac{r}{2}\sin\phi + \tanh^2\frac{r}{2}}, \tag{4}$$

the embedding coordinates can also be described by

$$x_1 = \sinh r\cos\phi, \qquad x_2 = \sinh r\sin\phi, \qquad x_3 = \cosh r. \tag{5}$$

The variables $(R = \tanh\frac{r}{2}, \phi)$ describe the hyperbolic polar coordinates in the Poincaré disc $\mathcal{D} = \{\eta = Re^{i\phi} : |\eta| < 1\}$ with metric induced from (2) as

$$ds^2 = 4\frac{dR^2 + R^2\, d\phi^2}{(1-R^2)^2} = dr^2 + \sinh^2 r\, d\phi^2. \tag{6}$$

Using (3) and (5) we are able to write the polar coordinates of the Poincaré disc in terms of the complex coordinates z and $\bar{z}$ of the Poincaré upper half-plane:

$$R = \frac{1 + i(z-\bar{z}) + |z|^2}{\sqrt{(1+z^2)(1+\bar{z}^2)}}, \qquad \tan\phi = \frac{|z|^2 - 1}{z+\bar{z}}. \tag{7}$$

We can immediately conclude that the complex variables η and z are related to each other by the following anti-Möbius transformation:

$$\eta = i\frac{\bar{z}+i}{\bar{z}-i}, \tag{8}$$

and this means that the embedding coordinates can also be described in terms of the complex variable corresponding to the Poincaré disc as

$$x_1 = \frac{\eta+\bar{\eta}}{1-|\eta|^2}, \qquad x_2 = i\frac{\eta-\bar{\eta}}{1-|\eta|^2}, \qquad x_3 = \frac{1+|\eta|^2}{1-|\eta|^2}. \tag{9}$$

3 Fuzzy AdS_2

The positive principal discrete representation of the $SU(1,1)$ with the generators J_1, J_2 and J_3, and with commutation relations

$$[J_i, J_j] = iC_{ij}{}^k J_k, \tag{10}$$

is realized in the Hilbert space through

$$\mathcal{D}^+(j) = \{|j,m\rangle : \ j>0, \ m=j, \ j+1, \ j+2, \ldots\} \tag{11}$$

as below:

$$\begin{aligned} J_+|j,m\rangle &= \sqrt{(m+j)(m-j+1)}|j,m+1\rangle, \\ J_-|j,m\rangle &= \sqrt{(m-j)(m+j-1)}|j,m-1\rangle, \\ J_3|j,m\rangle &= m|j,m\rangle, \end{aligned} \tag{12}$$

where $J_\pm = J_1 \pm iJ_2$. The state $|j,j\rangle$ has the lowest weight j in the Hilbert space $\mathcal{D}^+(j)$. Indeed, the irreps are determined by the eigenvalues of the Casimir operator $\mathcal{C} = J_1^2 + J_2^2 - J_3^2$,

$$\mathcal{C}|j,m\rangle = -j(j-1)|j,m\rangle. \tag{13}$$

To define the fuzzy AdS_2, the coordinates x_i's of (1) are promoted to play the role of the generators of $su(1,1)$, in some unitary representations

$$[x_i, x_j] = i\alpha C_{ij}{}^k x_k. \tag{14}$$

The structure constants $C_{ij}{}^k$ are determined as $C_{ij}{}^k = \eta^{kl} C_{ijl}$, in which $C_{123} = 1$ and C_{ijk}'s are completely antisymmetric with respect to the indices of adjustment. These structure constants $C_{ij}{}^k$ satisfy the following relations:

$$C_{im}{}^k \eta^{ij} C_{jl}{}^n = \eta_m{}^n \eta_l{}^k - \eta_{ml}\eta^{kn}. \tag{15}$$

The value for the Planck constant α is determined by the Casimir operator (1) as $\alpha = \frac{1}{\sqrt{j(j-1)}}$. The positive integer j which controls the strength of non-commutativity, plays the role of the angular momentum in the unitary irreducible j-representation space. The non-commutative relation (14) together with the embedding (1), which allocates a j-irrep space to the Casimir operator, defines the fuzzy AdS_2.

For a given positive integer j, let ($\mathcal{A}_{2j}$,$\mathcal{L}_{2j}$) denote the space of all analytic functions of coordinates and derivations on fuzzy AdS_2, respectively. The algebra $\mathcal{A}_{2j}$ of the fuzzy AdS_2 is generated by operators x_i and its elements acting on the infinite-dimensional Hilbert space $\mathcal{D}^+(j)$ spanned by a set of orthonormal bases $\{|j,m\rangle\}_{m=j}^\infty$. The number of independent functions over fuzzy AdS_2 is infinite since the matrix generators x_i that realize a j-irrep of the algebra $su(1,1)$ are of the infinite order. The generators L_i of $\mathcal{L}_{2j}$ are defined by the adjoint action of x_i on the space $\mathcal{A}_{2j}$:

$$\frac{1}{\alpha} ad_{x_i} x_j = \frac{1}{\alpha}[x_i, x_j] =: L_i x_j. \tag{16}$$

This induces the commutation relations of the Lie algebra $su(1,1)$ on $\mathcal{L}_{2j}$

$$[L_i, L_j] = iC_{ij}{}^k L_k. \tag{17}$$

The bigger algebra ($\mathcal{A}_{2j}, \mathcal{L}_{2j}$) includes the commutation relations (14) and (17), as well as

$$[L_i, x_j] = iC_{ij}{}^k x_k. \tag{18}$$

The j-irrep for the bases of the algebra $\mathcal{A}_{2j}$ may be represented as $x_k = \alpha J_k$ and hence (1) and (14) can be formed using (13) and (10), respectively. α vanishes in the limit $j \to \infty$, which implies the commutative limit.

4 The Weyl–Heisenberg inequality and a weaker type than that

To see how we can go from the fuzzy AdS_2 to the Poincaré upper half-plane and the Poincaré disc coordinates, it is helpful in this section to remember the theory of the generalized Weyl–Heisenberg inequality of Ref. [20]. Let the observables A and B be two self-adjoint operators in a complex and separable Hilbert space $\mathcal{H}$ that do not necessarily commute with each other. The interest is to find the positive lower bounds for the product of their variances independent of the state of the system. For this, set a general linear combination of the two operators and the identity operator:

$$\Theta_\lambda = A + (\lambda_1 + i\lambda_2)B + (\lambda_3 + i\lambda_4)I, \tag{19}$$

where λ_i's are real constants. The following steps have been simply taken from [20]. The inequality

$$\begin{aligned} &\langle\psi|\Theta_\lambda^\dagger \Theta_\lambda|\psi\rangle \\ &\quad= \langle A^2\rangle + 2\lambda_3\langle A\rangle + (\lambda_1^2+\lambda_2^2)\langle B^2\rangle \\ &\qquad + 2(\lambda_1\lambda_3+\lambda_2\lambda_4)\langle B\rangle + \lambda_1\langle\{A,B\}\rangle + i\lambda_2\langle[A,B]\rangle + \lambda_3^2 \\ &\qquad + \lambda_4^2 = \left[\left(\lambda_1 + \frac{\langle\{A,B\}\rangle + 2\lambda_3\langle B\rangle}{2\langle B^2\rangle}\right)^2\right. \\ &\qquad \left. + \left(\lambda_2 + \frac{i\langle[A,B]\rangle + 2\lambda_4\langle B\rangle}{2\langle B^2\rangle}\right)^2\right]\langle B^2\rangle \\ &\qquad + \left[\left(\lambda_3 + \frac{2\langle A\rangle\langle B^2\rangle - \langle\{A,B\}\rangle\langle B\rangle}{2(\Delta B)^2}\right)^2\right. \\ &\qquad \left. + \left(\lambda_4 - \frac{i\langle[A,B]\rangle\langle B\rangle}{2(\Delta B)^2}\right)^2\right]\frac{(\Delta B)^2}{\langle B^2\rangle} \\ &\qquad + \langle A^2\rangle + \frac{\langle[A,B]\rangle^2 - \langle\{A,B\}\rangle^2}{4\langle B^2\rangle} \\ &\qquad + \frac{\langle[A,B]\rangle^2\langle B\rangle^2 - (2\langle A\rangle\langle B^2\rangle - \langle\{A,B\}\rangle\langle B\rangle)^2}{4\langle B^2\rangle(\Delta B)^2} \\ &\quad \geq 0 \end{aligned} \tag{20}$$

holds for any unit state $\psi \in \mathcal{H}$ in the intersections $\text{dom}(A) \cap \text{dom}(B)$ of the domains of A and B. Here $\langle\cdot|\cdot\rangle$ denotes the inner product of $\mathcal{H}$. For an operator B, $\langle B\rangle$ represents the expectation value and $\Delta B=\sqrt{\langle B^2\rangle-\langle B\rangle^2}$ is the standard deviation with respect to a given state ψ. It is also obvious that the terms $\langle\{A,B\}\rangle$ and $i\langle[A,B]\rangle$ are real valued. To obtain the lower bound in (20) we take the fact that the two terms involving brackets are nonnegative, and this allows us to set the values of $\boldsymbol{\lambda}=(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ such that both of them tend to zero:

$$\begin{aligned}
\bar\lambda_1&=\frac{-1}{2\langle B^2\rangle}\big(\langle\{A,B\}\rangle-2\langle A\rangle\langle B\rangle\big)\left(1+\frac{\langle B\rangle^2}{2(\Delta B)^2}\right),\\
\bar\lambda_2&=\frac{-i\langle[A,B]\rangle}{2(\Delta B)^2},\\
\bar\lambda_3&=-\langle A\rangle+\frac{\langle B\rangle}{2(\Delta B)^2}\big(\langle\{A,B\}\rangle-2\langle A\rangle\langle B\rangle\big),\\
\bar\lambda_4&=\frac{i\langle B\rangle\langle[A,B]\rangle}{2(\Delta B)^2}.
\end{aligned}\tag{21}$$

The inequality (20) can now be expressed more simply as

$$(\Delta A)^2(\Delta B)^2\geq\frac{1}{4}\Big(\big(\langle\{A,B\}\rangle-2\langle A\rangle\langle B\rangle\big)^2+\big|\langle[A,B]\rangle\big|^2\Big).\tag{22}$$

This inequality, which is weaker than the Weyl–Heisenberg inequality, is saturated for every unit state $\psi\in\mathcal{H}$ and our aim is to obtain the smallest value of $(\Delta A)(\Delta B)$. The squeezed states are those which verify the equality case of the inequality (22) and can be described by

$$\Theta_{\bar{\boldsymbol\lambda}}|\psi\rangle=0.\tag{23}$$

A comparison with the coherent states is also made in continuation of this report. The smallest value for the right hand side of (22) is obtained when its first term is equal to zero, i.e.

$$\langle\{A,B\}\rangle=2\langle A\rangle\langle B\rangle.\tag{24}$$

If the state ψ is such that (24) is satisfied then the inequality (22) is reduced to the Weyl–Heisenberg uncertainty relation

$$\Delta A\Delta B\geq\frac{1}{2}\big|\langle[A,B]\rangle\big|,\tag{25}$$

and, consequently, $\bar\lambda_1$ and $\bar\lambda_3$ are calculated as

$$\bar\lambda_1^0=0,\qquad\bar\lambda_3^0=-\langle A\rangle.\tag{26}$$

Therefore, for such a ψ we get

$$\Theta_{\bar{\boldsymbol\lambda}^0}|\psi\rangle=\left(A-\langle A\rangle+\frac{\langle[A,B]\rangle}{2(\Delta B)^2}\big(B-\langle B\rangle\big)\right)|\psi\rangle,\tag{27}$$

and it immediately results that $\langle\psi|\Theta_{\bar{\boldsymbol\lambda}^0}|\psi\rangle=0$. Also, (25) is reduced to the equality case when the state ψ satisfies the relation

$$\Theta_{\bar{\boldsymbol\lambda}^0}|\psi\rangle=0\tag{28}$$

and therefore the Weyl–Heisenberg inequality is minimized by such a ψ, i.e. the so-called coherent states.

5 The generalized squeezed states on the fuzzy AdS_2

Now, the question is whether or not one can find the generalized coherent states (preferably from the generalized squeezed states) in order to simultaneously minimize the three uncertainty relations among non-commuting variables x_1, x_2 and x_3 of fuzzy AdS_2:

$$\begin{aligned}
&\Delta x_1\Delta x_2=\frac{1}{2}\big|\langle[x_1,x_2]\rangle\big|,\qquad\Delta x_2\Delta x_3=\frac{1}{2}\big|\langle[x_2,x_3]\rangle\big|,\\
&\Delta x_3\Delta x_1=\frac{1}{2}\big|\langle[x_3,x_1]\rangle\big|.
\end{aligned}\tag{29}$$

The steps of the computations in this section follow closely the steps of Ref. [20]. We will show that (29) cannot be satisfied simultaneously by an appropriate state, therefore, we look for the states (the so-called generalized squeezed states) which simultaneously satisfy the three following relations:

$$\begin{aligned}
(\Delta x_1)^2(\Delta x_2)^2&=\frac{1}{4}\Big(\big(\langle\{x_1,x_2\}\rangle-2\langle x_1\rangle\langle x_2\rangle\big)^2-\langle[x_1,x_2]\rangle^2\Big),\\
(\Delta x_2)^2(\Delta x_3)^2&=\frac{1}{4}\Big(\big(\langle\{x_2,x_3\}\rangle-2\langle x_2\rangle\langle x_3\rangle\big)^2-\langle[x_2,x_3]\rangle^2\Big),\\
(\Delta x_3)^2(\Delta x_1)^2&=\frac{1}{4}\Big(\big(\langle\{x_3,x_1\}\rangle-2\langle x_3\rangle\langle x_1\rangle\big)^2-\langle[x_3,x_1]\rangle^2\Big).
\end{aligned}\tag{30}$$

In accordance with (19), it is necessary to use any of the pairs (x_1,x_2), (x_2,x_3) and (x_3,x_1) instead of (A,B) in order to define the three operators, respectively:

$$\begin{aligned}
&M_{12}=x_1+\mu_{12}x_2+\tau_{12},\qquad M_{23}=x_2+\mu_{23}x_3+\tau_{23},\\
&M_{31}=x_3+\mu_{31}x_1+\tau_{31}.
\end{aligned}\tag{31}$$

By relying on (28), the generalized coherent states satisfy simultaneously the three conditions as stated below

$$M_{12}|\psi\rangle=0,\qquad M_{23}|\psi\rangle=0,\qquad M_{31}|\psi\rangle=0.\tag{32}$$

For this purpose, it is necessary to note that from (26) it follows that μ_{12}, μ_{23} and μ_{31} are pure imaginary constants. The following relations in turn follow from (32):

$$\begin{aligned}&\big(\mu_{31}[M_{12},M_{23}]+[M_{23},M_{31}]\big)|\psi\rangle=0,\\&\big(\mu_{23}[M_{31},M_{12}]+[M_{12},M_{23}]\big)|\psi\rangle=0,\\&\big(\mu_{12}[M_{23},M_{31}]+[M_{31},M_{12}]\big)|\psi\rangle=0.\end{aligned}\tag{33}$$

Consequently the following three independent relations are obtained by using $su(1,1)$ commutation relations (14), similar to what has happened to the fuzzy sphere in [20]:

$$\begin{aligned}&(1+\mu_{12}\mu_{23}\mu_{31})x_1|\psi\rangle=0,\\&(1+\mu_{12}\mu_{23}\mu_{31})x_2|\psi\rangle=0,\\&(1+\mu_{12}\mu_{23}\mu_{31})x_3|\psi\rangle=0.\end{aligned}\tag{34}$$

Therefore, at least one of the relations

$$x_k|\psi\rangle=0\quad\text{for } k=1,2,3,\qquad \mu_{12}\mu_{23}\mu_{31}=-1,\tag{35}$$

should be satisfied. As the first relation is inconsistent with (1), we conclude that the second relation of (35) must be satisfied. This in turn, implies that the three constants μ_{12}, μ_{23} and μ_{31} cannot be pure imaginary numbers simultaneously. This means that the Weyl–Heisenberg uncertainty relations between x_1 and x_2, x_2 and x_3 as well as x_3 and x_1 cannot be simultaneously minimized by an appropriate generalized coherent state.

In what follows, by using (23) we investigate the generalized squeezed states as the ones which satisfy (32) and consider the fact that each of the parameters μ_{12}, μ_{23} and μ_{31} as well as τ_{12}, τ_{23} and τ_{31} can be complex-valued. Firstly, it must be noticed that using (21), we can get the expectation values of the non-commuting coordinates over such states as follows:

$$\begin{aligned}&\langle x_1\rangle=-\frac{\mathrm{Im}(\tau_{31})}{\mathrm{Im}(\mu_{31})},\qquad \langle x_2\rangle=-\frac{\mathrm{Im}(\tau_{12})}{\mathrm{Im}(\mu_{12})},\\&\langle x_3\rangle=-\frac{\mathrm{Im}(\tau_{23})}{\mathrm{Im}(\mu_{23})}.\end{aligned}\tag{36}$$

As usual, "Im" denotes the imaginary part of a complex number. Equations (32) for a general form of the generalized squeezed states $|\psi\rangle$ with the following expansion coefficients:

$$|\psi\rangle=\sum_{m=j}^{\infty}\psi_m|j,m\rangle,\tag{37}$$

are transformed to six relations as below

$$\frac{\psi_{j+1}}{\psi_j}=\frac{-2\tau_{12}}{\alpha(1+i\mu_{12})\sqrt{2j}},\tag{38}$$

$$\psi_{m+1}+\frac{2\tau_{12}}{\alpha(1+i\mu_{12})\sqrt{(m+j)(m-j+1)}}\psi_m+\frac{1-i\mu_{12}}{1+i\mu_{12}}\sqrt{\frac{(m-j)(m+j-1)}{(m+j)(m-j+1)}}\psi_{m-1}=0,\tag{39}$$

$$\frac{\psi_{j+1}}{\psi_j}=\frac{-2(\alpha\mu_{23}j+\tau_{23})}{i\alpha\sqrt{2j}},\tag{40}$$

$$\psi_{m+1}+\frac{2(\alpha\mu_{23}m+\tau_{23})}{i\alpha\sqrt{(m+j)(m-j+1)}}\psi_m-\sqrt{\frac{(m-j)(m+j-1)}{(m+j)(m-j+1)}}\psi_{m-1}=0,\tag{41}$$

$$\frac{\psi_{j+1}}{\psi_j}=\frac{-2(\alpha j+\tau_{31})}{\alpha\mu_{31}\sqrt{2j}},\tag{42}$$

$$\psi_{m+1}+\frac{2(\alpha m+\tau_{31})}{\alpha\mu_{31}\sqrt{(m+j)(m-j+1)}}\psi_m+\sqrt{\frac{(m-j)(m+j-1)}{(m+j)(m-j+1)}}\psi_{m-1}=0,\tag{43}$$

where we have used (12) for a given j. According to our discussions in Sect. 3, the Planck constant α, as a non-negative real parameter, controls the strength of the non-commutativity. By comparing (38) with (40) and (42), the parameters τ_{23} and μ_{31} satisfy the relations

$$\tau_{23}=-\alpha j\mu_{23}+\frac{i\tau_{12}}{1+i\mu_{12}},\tag{44}$$

$$\mu_{31}=\frac{(1+i\mu_{12})(\alpha j+\tau_{31})}{\tau_{12}}.\tag{45}$$

Multiplying (39) by $-(1+i\mu_{12})/(1-i\mu_{12})$ and then adding the obtained result to (43), also adding (41) to (43), and then, by comparing their results with each other, we get

$$\mu_{23}=\frac{-\tau_{12}}{\mu_{12}(1+i\mu_{12})(\alpha j+\tau_{31})}.\tag{46}$$

Subtracting relation (41) from (43) we obtain

$$\frac{\psi_m}{\psi_{m-1}}=\frac{-\alpha\sqrt{(m-j)(m+j-1)}}{\frac{\alpha m+\tau_{31}}{\mu_{31}}+i(\alpha m\mu_{23}+\tau_{23})},\quad m=j+1,j+2,\ldots.\tag{47}$$

Equation (47) for $m=j+1$ gives

$$\frac{\psi_{j+1}}{\psi_j}=\frac{-i\mu_{12}\sqrt{2j}(\alpha j+\tau_{31})}{\tau_{12}},\tag{48}$$

where we have used (45) and (46). Now, a comparison between (38) and (48) gives

$$\tau_{31}=-\alpha j+\frac{\tau_{12}^2}{i\alpha j\mu_{12}(1+i\mu_{12})}.\tag{49}$$

After substituting (47) in (41) and using the appropriate definitions mentioned above we get

$$\tau_{12} = -i\alpha j\sqrt{1+\mu_{12}^2}. \tag{50}$$

Finally, by defining the complex variable $\xi = \frac{\psi_{j+1}}{\psi_j}$ and substituting (50) in (38), the parameter μ_{12} is derived as

$$\mu_{12} = i\frac{\xi^2+2j}{\xi^2-2j}. \tag{51}$$

Now, it is not difficult to calculate the final form of the remaining parameters,

$$\begin{aligned}
\mu_{23} &= \frac{1}{\sqrt{1+\mu_{12}^2}} = \frac{-i}{2\sqrt{2j}}\frac{\xi^2-2j}{\xi},\\
\mu_{31} &= -\frac{\sqrt{1+\mu_{12}^2}}{\mu_{12}} = -2\sqrt{2j}\frac{\xi}{\xi^2+2j},\\
\tau_{12} &= 2j\sqrt{\frac{2}{j-1}}\frac{\xi}{\xi^2-2j},\\
\tau_{31} &= \frac{i\alpha j}{\mu_{12}} = \sqrt{\frac{j}{j-1}}\frac{\xi^2-2j}{\xi^2+2j},\\
\tau_{23} &= \frac{i\alpha j}{\mu_{31}} = \frac{-i}{2\sqrt{2(j-1)}}\frac{\xi^2+2j}{\xi}.
\end{aligned} \tag{52}$$

Also, the recursion relation (47) and its solution in terms of ξ are given by

$$\begin{aligned}
\frac{\psi_m}{\psi_{m-1}} &= \sqrt{\frac{m+j-1}{m-j}}\frac{\xi}{\sqrt{2j}},\\
\psi_m &= \sqrt{C_{m-j}^{m+j-1}}\left(\frac{\xi}{\sqrt{2j}}\right)^{m-j}\psi_j,
\end{aligned} \tag{53}$$

where C_{m-j}^{m+j-1} denotes the binomial coefficient. We apply the orthonormality relation $\langle j,m|j,m'\rangle = \delta_{mm'}$ to normalize the squeezed state $|\psi\rangle$ and obtain the normalization constant ψ_j with a constraint on the module of ξ,

$$\psi_j = \left(1-\frac{|\xi|^2}{2j}\right)^j, \quad \frac{|\xi|^2}{2j} < 1. \tag{54}$$

Consequently the squeezed states are derived as

$$|\xi\rangle = \left(1-\frac{|\xi|^2}{2j}\right)^j \sum_{m=j}^{\infty}\sqrt{C_{m-j}^{m+j-1}}\frac{\xi^{m-j}}{(2j)^{\frac{m-j}{2}}}|j,m\rangle. \tag{55}$$

Now we are in a position that, for a fixed irreducible representation with j, we can define a new complex variable as $\eta = \frac{\xi}{\sqrt{2j}}$ and derive a final form for the squeezed states on the Poincaré disc $\mathcal{D}$ as

$$\begin{aligned}
|\eta\rangle &= \left(1-|\eta|^2\right)^j \sum_{m=0}^{\infty}\sqrt{C_m^{m+2j-1}}\,\eta^m|j,m+j\rangle,\\
&|\eta| < 1.
\end{aligned} \tag{56}$$

In (55) and (56), the symbols $|\xi\rangle$ and $|\eta\rangle$ have been used instead of $|\psi\rangle$ for more appropriateness and in accordance with what is in vogue. It should be emphasized that these generalized squeezed states satisfy simultaneously the three uncertainty relations given in (30). Finally, from the orthonormality relation for the bases $|j,m\rangle$, it can be seen that the overlapping of two different squeezed states is given by the following non-vanishing expression:

$$\langle\eta|\eta'\rangle = \frac{(1-|\eta|^2)^j(1-|\eta'|^2)^j}{(1-\bar{\eta}\eta')^{2j}}, \tag{57}$$

which is converted to the relation $\langle\eta|\eta\rangle = 1$ for $\eta = \eta'$.

6 Poincaré coordinates from the squeezed states on fuzzy AdS_2

This section is devoted to analyze expectation values of the non-commuting variables x_1, x_2 and x_3 of fuzzy AdS_2 over the generalized squeezed states. As an example, the expectation value of x_1 can be calculated as

$$\begin{aligned}
\langle x_1\rangle &= \frac{1}{\sqrt{j(j-1)}}\left(1-\frac{|\xi|^2}{2j}\right)^{2j}\\
&\quad\times\sum_{m,m'=j}^{\infty}\sqrt{C_{m-j}^{m+j-1}C_{m'-j}^{m'+j-1}}\frac{\xi^{m-j}\bar{\xi}^{m'-j}}{(2j)^{\frac{m+m'-2j}{2}}}\\
&\quad\times\langle j,m'|J_1|j,m\rangle\\
&= j\sqrt{\frac{2}{j-1}}\frac{\xi+\bar{\xi}}{2j-|\xi|^2} = \sqrt{\frac{j}{j-1}}\frac{\eta+\bar{\eta}}{1-|\eta|^2}.
\end{aligned} \tag{58}$$

Similarly, we get

$$\langle x_2\rangle = ij\sqrt{\frac{2}{j-1}}\frac{\xi-\bar{\xi}}{2j-|\xi|^2} = i\sqrt{\frac{j}{j-1}}\frac{\eta-\bar{\eta}}{1-|\eta|^2}, \tag{59}$$

$$\langle x_3\rangle = \sqrt{\frac{j}{j-1}}\frac{2j+|\xi|^2}{2j-|\xi|^2} = \sqrt{\frac{j}{j-1}}\frac{1+|\eta|^2}{1-|\eta|^2}. \tag{60}$$

Now, it is easy from the first and the third relations of (32) to conclude that

$$\begin{aligned}
&\langle x_1\rangle + \mu_{12}\langle x_2\rangle + \tau_{12} = 0,\\
&\langle x_3\rangle + \mu_{31}\langle x_1\rangle + \tau_{31} = 0.
\end{aligned} \tag{61}$$

In order to see the consistency of the theory, we put the values $\mu_{12}, \tau_{12}, \mu_{31}, \tau_{31}$ of (51) and (52) as well as $\langle x_1 \rangle$ of (58) into (61) and again obtain the results (59) and (60) for the expectation values $\langle x_2 \rangle$ and $\langle x_3 \rangle$, respectively. Also, the expectation values can directly be calculated from (36). With the help of (8), the expectation values of the embedding coordinates for the fuzzy AdS_2 can now be calculated in terms of the complex coordinates z and $\bar{z}$ of the Poincaré upper half-plane:

$$\langle x_1 \rangle = -i\sqrt{\frac{j}{j-1}}\frac{z+\bar{z}}{z-\bar{z}}, \qquad \langle x_2 \rangle = i\sqrt{\frac{j}{j-1}}\frac{1-|z|^2}{z-\bar{z}}, \qquad \langle x_3 \rangle = i\sqrt{\frac{j}{j-1}}\frac{1+|z|^2}{z-\bar{z}}. \tag{62}$$

As an immediate result of these conclusions we see that the expectation values satisfy the following relation corresponding to the classical relation (1):

$$\langle x_1 \rangle^2 + \langle x_2 \rangle^2 - \langle x_3 \rangle^2 = -\frac{j}{j-1}. \tag{63}$$

It is now straightforward to show that the embedding coordinates (3) and (9) for the classical AdS_2 follows from the expectation values of the embedding coordinates for the fuzzy AdS_2, i.e. (62) as well as (58), (59) and (60), by the limiting process $j \to \infty$, respectively. It is clear that, in the limit of j approaching infinity, the constraint (63) on the expectation values of the fuzzy embedding coordinates is also converted to the embedding equation of the classical AdS_2.

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