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Dirac operators on the fuzzy AdS_2 with the spins $\frac{1}{2}$ and 1

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It is shown here how the pseudo chirality and Dirac operators with the spins $\frac{1}{2}$ and 1 on the commutative and fuzzy AdS_2 should be constructed. The finite-dimensional and nonunitary representations of $SU(1, 1)$ carrying the spin degrees of freedom $\frac{1}{2}$ and 1 are used for the Dirac fields on commutative and fuzzy AdS_2. In the fuzzy case, an explicit description of pseudo generalization of the Ginsparg-Wilson algebra is used to construct projective modules. The projector couplings left angular momentum and spin on the fuzzy AdS_2 are used to produce minimum total angular momenta. They are realized by the first two and three representations of the total angular momentum for the spins $\frac{1}{2}$ and 1, respectively. The pseudo projectors, the pseudo chirality, and Dirac operators with the spins $\frac{1}{2}$ and 1 on the fuzzy AdS_2 tend to their corresponding operators in the commutative limit. © *2011 American Institute of Physics*. [doi:10.1063/1.3653480]

I. INTRODUCTION

Nowadays, one of the most important aims in theoretical physics is to use noncommutative geometry in order to construct appropriate models for the space-time structure. The noncommutativity of the space coordinates leads to the nonexistence of concept of the "point." Thus, all attention is concentrated to reformulate the geometry of a space in terms of the commutative algebras and modules of smooth functions and then their generalization to noncommutative analogs. Connes' formulation of noncommutative geometry based on the real spectral triple is an outstanding approach to get very brilliant results.[1,2]

Dirac K-cycle or spectral triple is the most important ingredient in noncommutative geometry. An even K-cycle $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \Gamma)$ has the following properties:

- $*$ algebra $\mathcal{A}$ is faithfully represented by bounded operators on a separable Hilbert space $\mathcal{H}$.
- $\mathcal{D}$ is a self-adjoint operator on $\mathcal{H}$; $[\mathcal{D}, a]$ is a bounded operator for all $a \in \mathcal{A}$, and $(1 + \mathcal{D})^{-1}$ is a compact operator on $\mathcal{H}$.
- Γ is a $\mathbb{Z}_2$ grading self-adjoint (chirality) operator on $\mathcal{H}$ such that $\Gamma^2 = 1$, $\{\Gamma, \mathcal{D}\} = 0$ and $[\mathcal{D}, a] = 0$ for all $a \in \mathcal{A}$.

These requirements do not uniquely fix the operators Γ and $\mathcal{D}$. For the odd-dimensional manifolds, there are no chirality operators and in this case the Dirac operators describe only differential structures. The chirality operators exist for the even-dimensional manifolds, and they $\mathbb{Z}_2$-grade the space of spinors. In this case, both of the metric and differential structures are described by the Dirac operators. For example, the distance formula is given in terms of $\mathcal{D}$, and in order to do the integration over manifold, we need to know the spectrum of $\mathcal{D}$. In fact, the geometry and topology of a manifold is uniquely determined by the (even) K-cycle $(\mathcal{A}, \mathcal{H}, \mathcal{D}, (\Gamma))$.

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Noncommutative spaces in physics were initially introduced to regularize the divergences encountered in field theory by geometric quantization of the manifolds. They possess a rich mathematical structure which can be elegant and highly important for the solutions of various problems. When the effective constant of quantization tends to zero, the fuzzy manifold, as a quantum representation of classical manifold, is transformed to the classical one. Many systems of interest in physics have configuration spaces with the form of simple or semi-simple Lie groups. $SO(3)$ and $SL(2,\mathbb{R})$ are the symmetry groups for two of the most famous surfaces, the so-called the sphere $S^2 = SU(2)/U(1)$ and hyperbolic plane $AdS_2 = SU(1,1)/U(1)$. Many studies have been made on the fuzzy version of the first since it is compact and therefore fuzzy sphere becomes a finite-dimensional matrix algebra on which the Lie group $SO(3)$ acts in simple ways. Fuzzy sphere, which can be identified with the two-dimensional gravity of the two-sphere, was first introduced by Madore in the context of noncommutative geometry,[3] and was later used in various different physical applications.[4–6] A different method cooperating spectral triples has been used to study the spectrum of Dirac operator, differential calculus, Chern character and topological charges on the fuzzy sphere.[7–12] Furthermore, there are a few studies that have examined different methods to formulate the noncommutative analog of a noncompact surface with constant negative curvature. The fuzzy AdS_2 is appeared as a descendant of the 4-dimensional extremal black hole with unusual quantum properties. For this model, the quantum analogue of the Laplace equation is solved and a strong consistency check is provided by calculating three point vertices and demonstrating the conventional $1/N$ expansion as well as non-perturbative effects in large N.[13] The fuzzy AdS_2 model is also applied to derive holography from space-time noncommutativity.[14] Besides, in Ref. 15, the Landau problem has been formulated on the fuzzy AdS_2 and there it has been shown that the degeneracy of the discrete spectrum in the commutative AdS_2 is lifted in the fuzzy case. The Watamuras' approach along with pseudo-Hermiticity have been used to study the spectrum of Dirac operator on the fuzzy AdS_2 too.[16]

Instead of being self-adjoint as in the case of compact manifold, the chirality and Dirac operators turn out to be pseudo Hermitian on the (fuzzy) AdS_2. The Hermiticity framework has recently been extended to pseudo-Hermiticity with the help of a metric operator derived from the inner product.[17,18] It has been shown that pseudo-Hermiticity is equivalent to the presence of anti-linear symmetries.[19] Furthermore, it has been proven that the criterion for the pseudo-Hermiticity of the non-Hermitian operators is the spectra to appear as real or complex-conjugate pairs. Thus, based on the concept of pseudo-Hermiticity we can consider the pseudo chirality and Dirac operators on (fuzzy) AdS_2.

In this paper, our method is based on the Balachandran and Padmanabhan's approach to the fuzzy 2-sphere.[20] Along the lines of,[21–25] it is interesting to see how the chirality and Dirac operators are modified on fuzzy AdS_2. The paper has been organized as follows: Sec. II briefly describes construction of the free and projective modules on the commutative and fuzzy AdS_2. In Sec. III, the 2×2 and 3×3 Hermitian generators of the spins $\frac{1}{2}$ and 1 are used to construct the nonunitary representations of $SU(1,1)$ and the pseudo generalization of the Ginsparg-Wilson algebra. Section IV is devoted to introducing the fuzzy non-Hermitian matrix algebra $\mathcal{A}_l$ and the fuzzy geometry of AdS_2. The form of the pseudo chirality and Dirac operators and the projector coupling of the spins $\frac{1}{2}$ and 1 with angular momentum l on the usual AdS_2 is recalled in Sec. V. Section VI expresses how to use the first two and three representation subspaces of the total angular momentum on the fuzzy AdS_2 to construct the pseudo fuzzy operators with the spins $\frac{1}{2}$ and 1, respectively. The fuzzy versions of the pseudo chirality and Dirac operators and projectors are transformed to the commutative limit, similar to what has been done in Ref. 16.

II. THE MODULE CONSTRUCTION

Consider the $U(1)$ Hopf principal fibration π of the total space AdS_3 over AdS_2 (for details, see Refs. 16 and 26):

$$U(1) \overset{\text{rightU(1)}-\text{action}}{\hookrightarrow} AdS_3 \xrightarrow{\pi} AdS_2. \tag{2.1}$$

Let $\mathcal{B}_{\mathbb{C}} = C^\infty(AdS_3, \mathbb{C})$ and $\mathcal{A}_{\mathbb{C}} = C^\infty(AdS_2, \mathbb{C})$ denote the commutative algebras of $\mathbb{C}$-valued smooth functions on the total space AdS_3 and the base space AdS_2 under point-wise multiplication, respectively. The elements of $\mathcal{B}_{\mathbb{C}}$ are classified into the right modules,

$$C^{\infty}_{(n)}(AdS_3,\mathbb{C})=\{\varphi:AdS_3\to\mathbb{C},\quad \varphi(p.w)=w^{-n}.\varphi(p),\ \ \forall p\in AdS_3,\ \forall w\in U(1)\},\qquad (2.2)$$

over the pull-back of the $\mathcal{A}_{\mathbb{C}}$, with n as non-negative integer numbers. The Serre-Swan's theorem[27] states that for a compact smooth manifold, there is a complete equivalence between the category of smooth vector bundles and the category of finitely generated projective modules. This equivalence is extended in Ref. 28 to the case of noncompact manifolds. Therefore, for the associated vector bundle

$$E^{(n)}=AdS_3\times_{U(1)}\mathbb{C}\xrightarrow{\ \pi\ }AdS_2,\qquad (2.3)$$

right $\mathcal{A}_{\mathbb{C}}$-module of sections $\Gamma^{\infty}(AdS_2,E^{(n)})$ - isomorphic with $C^{\infty}_{(n)}(AdS_3,\mathbb{C})$ - is equivalent to the image in the free module $(\mathcal{A}_{\mathbb{C}})^{n+1}=C^{\infty}(AdS_2,\mathbb{C})\otimes\mathbb{C}^{n+1}$ of a projector P: $\Gamma^{\infty}(AdS_2,E^{(n)})=P(\mathcal{A}_{\mathbb{C}})^{n+1}$. The projector P is invariant under the similarity transformation induced by the $U(1)$ right action. Thus, entries of the projector P are elements of the algebra $\mathcal{A}_{\mathbb{C}}$, and it is a Λ-pseudo Hermitian operator of the rank 1 over $\mathbb{C}$,

$$P\in\mathbb{M}_{n+1}(\mathcal{A}_{\mathbb{C}}),\qquad P^2=P,\qquad P^{\dagger}=\Lambda P\Lambda^{-1},\qquad \mathrm{Tr(P)}=1,\qquad (2.4)$$

with Λ as a Hermitian, involutory, and unitary matrix. In case of compact manifolds, such as the sphere S^2, the property of pseudo-Hermiticity is reduced to Hermiticity.[29] For the right $\mathcal{A}_{\mathbb{C}}$-module of sections $\Gamma^{\infty}(AdS_2,E^{(n)})$, there exist $n+1$ complementary pseudo projectors $P_1, P_2, \cdots, P_{n+1}$ having the same rank 1 which are mutually orthoghonal in the ordinary sense. Therefore, the free module $(\mathcal{A}_{\mathbb{C}})^{n+1}$ can be written as a direct sum of the projective $\mathcal{A}_{\mathbb{C}}$-modules,

$$(\mathcal{A}_{\mathbb{C}})^{n+1}=\oplus_{i=1}^{n+1}P_i\,(\mathcal{A}_{\mathbb{C}})^{n+1}.\qquad (2.5)$$

We will confine the value of integer number n in this paper to $2s$ in which s is the spin number.

In Secs. III–VI, we will show how the property of Λ-pseudo-Hermiticity allows us to study pseudo projectors with the spins $s=\frac{1}{2}$ and 1, and to express the free modules in terms of the projective ($\mathcal{A}_l$-) $\mathcal{A}_{\mathbb{C}}$-modules on the (fuzzy) AdS_2. The commutative algebra $\mathcal{A}_{\mathbb{C}}$ is substituted by the noncommutative matrix algebra $\mathbb{M}_{2l+1}(\mathbb{C})$ when we want to pass from the usual to a noncommutative matrix geometry.[30] One of the subspaces of $\mathbb{M}_{2l+1}(\mathbb{C})$ is chosen such that its associated derivations form a proper subalgebra of the Lie algebra $\mathrm{Der}(\mathbb{M}_{2l+1}(\mathbb{C}))$. So, we will choose the subspace and its derived algebra, denoted by $\mathcal{A}_l$, as the Λ-pseudo Hermitian generators of the $su(1,1)$ Lie algebra to avoid to deal with the infinite-dimensional matrices. $\mathcal{A}_l$ is a bimodule since it carries the left- and right-regular representations of the fuzzy algebra. Again for every $(2s+1)\times(2s+1)$ nonunitary representations of $SU(1,1)$, there are $2s+1$ complementary orthogonal and Λ-pseudo-Hermiticity projectors $P_1, P_2, \cdots, P_{2s+1}$ so that generalization of (2.5) to the fuzzy AdS_2 can be made,

$$(\mathcal{A}_l)^{2s+1}=\oplus_{i=1}^{2s+1}P_i\,(\mathcal{A}_l)^{2s+1},\qquad (2.6)$$

where $(\mathcal{A}_l)^{2s+1}=\mathcal{A}_l\otimes\mathbb{C}^{2s+1}$ and $P_i\,(\mathcal{A}_l)^{2s+1}$ are free and projective modules, respectively.

III. NONUNITARY REPRESENTATIONS OF *SU*(1, 1) AND PSEUDO GENERALIZATION FOR THE GINSPARG-WILSON ALGEBRA

The pseudo version of the Ginsparg-Wilson algebra $\mathcal{A}$ is defined by two Λ-pseudo Hermitian involutions Γ and Γ'

$$\mathcal{A}=\left\{\Gamma,\Gamma':\ \ \Gamma^2=\Gamma'^2=I,\ \ \Gamma^{\dagger}=\Lambda\Gamma\Lambda^{-1},\ \ \Gamma'^{\dagger}=\Lambda\Gamma'\Lambda^{-1}\right\},\qquad (3.1)$$

in which the operator Λ is Hermitian, involutory, and unitary

$$\Lambda^{\dagger}=\Lambda,\qquad \Lambda^2=I,\qquad \Lambda^{-1}=\Lambda^{\dagger}.\qquad (3.2)$$

I is the identity element of $\mathcal{A}$. The generator Γ of the Ginsparg-Wilson algebra should not be confused with chirality operator of the Dirac K-cycle. It is obvious that adjoint of the generators of pseudo Ginsparg-Wilson algebra are an involutive pair, too. Here, Γ, Γ', and Λ are considered to be 2×2

and 3×3 matrices for spins $\frac{1}{2}$ and 1, respectively. Λ establishes the relation between eigenvectors Γ and $\Gamma^\dagger$ (also Γ' and $\Gamma'^\dagger$), and it provides the possibility to make the nonunitary representation of the $SU(1, 1)$ with the non-Hermitian matrix generators Σ_1, Σ_2, and Σ_3,

$$[\Sigma_i, \Sigma_j] = iC_{ij}{}^k \Sigma_k. \tag{3.3}$$

The structure constants $C_{ij}{}^k$ are determined as $C_{ij}{}^k = \eta^{kl} C_{ijl}$, in which $C_{123}=1$ and C_{ijk}'s are completely antisymmetric with respect to the indices of adjustment. Also, the Minkowskian metric $\eta^{ij} = \eta_{ij} = \mathrm{diag}(1, 1, -1)$ raises and lowers the indices. It is easy to show that the structure constants $C_{ij}{}^k$ satisfy the following relations:

$$C_{im}{}^k \eta^{ij} C_{jl}{}^n = \eta_m{}^n \eta_l{}^k - \eta_{ml}\eta^{kn}. \tag{3.4}$$

The generators of $su(1, 1)$ are pseudo Hermitian with respect to Λ

$$\Sigma_i^\dagger = \Lambda \Sigma_i \Lambda^{-1}, \tag{3.5}$$

and consequently, the commutation relations (3.3) are also satisfied by $\Sigma_1^\dagger$, $\Sigma_2^\dagger$, and $\Sigma_3^\dagger$. This induces the Λ-pseudo-Hermiticity structure of the algebra on the Hilbert space $\mathcal{H}$ on which $su(1, 1)$ acts.

As pointed out above, the $su(1, 1)$ commutation relations (3.3) are not closed with respect to $\dagger$, and this means that Σ_1, Σ_2 and Σ_3 will describe the spins $\frac{1}{2}$ and 1 for the Dirac operator constructed by the pseudo Ginsparg-Wilson algebra $\mathcal{A}$. An appropriate inner product between the elements of the Hilbert space $\mathcal{H}$ is defined below. We consider the linear operator $u : \mathcal{H} \to \mathcal{H}$ as an arbitrary element of the Lie algebra $su(1, 1)$ acting on the separable Hilbert space $\mathcal{H}$. Indeed, $\mathcal{H}$ as a Kerin space is equipped with a natural indefinite inner product "*" as $\Psi * \Phi = \overline{\Psi}\Phi$. Ψ and Φ are two arbitrary elements of $\mathcal{H}$, and " $-$ " denotes the Λ-adjoint: $\overline{\Psi} = \Psi^\dagger \Lambda$. So, one concludes immediately that all the generators belonging to $su(1, 1)$ are Λ-pseudo Hermitian with respect to the inner product "*".

A. Spin $\frac{1}{2}$

Every representation of the spin operators can analytically be continued to a finite dimensional, necessarily nonunitary representation of the $SU(1, 1)$ group. The generators corresponding to the lowest dimensional nonunitary representation of the $SU(1, 1)$, i.e., 2×2 matrices, can be constructed by the Pauli matrices σ_1, σ_2, and σ_3 as

$$\Sigma_1 = \frac{i}{2}\sigma_1, \qquad \Sigma_2 = \frac{i}{2}\sigma_2, \qquad \Sigma_3 = \frac{1}{2}\sigma_3. \tag{3.6}$$

It is straightforward to show that

$$\Sigma_i \Sigma_j = -\frac{1}{4}\eta_{ij} I + \frac{1}{2} i C_{ij}{}^k \Sigma_k. \tag{3.7}$$

So, the commutation relations (3.3) of the Lie algebra $su(1, 1)$ and the Clifford algebra $\{\Sigma_i, \Sigma_j\} = -\frac{1}{2}\eta_{ij} I$ immediately follow from (3.7). In this case, we have $\Lambda = \Sigma_3$, that is,

$$\Sigma_i^\dagger = \Sigma_3 \Sigma_i \Sigma_3^{-1}. \tag{3.8}$$

This means that the $su(1, 1)$ Lie algebra (3.3) and the identity relation (3.7) together with the Clifford algebra are not invariant under $\dagger$. Similar relations are satisfied by the generators $\Sigma_i^\dagger$. It is well known that the Pauli matrices are used to describe the spin $\frac{1}{2}$ operator with the $su(2)$ commutation relations.

B. Spin 1

In this case, the Λ-self adjointness property for traceless 3×3 matrices

$$\Sigma_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \qquad \Sigma_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \Sigma_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.9}$$

follows from the self-adjoint, involutory and unitary operator $\Lambda = \eta$,

$$\Sigma_i^\dagger = \eta \Sigma_i \eta^{-1}. \tag{3.10}$$

They constitute a nonunitary 3×3 representation of the $SU(1, 1)$ with the following identities:

$$\Sigma_i \Sigma_j = -\frac{2}{3}\eta_{ij} + \frac{i}{2} C_{ij}{}^m \Sigma_m - \eta_i{}^m Q_{mj}, \tag{3.11}$$

$$\Sigma_i \Sigma_k \Sigma_j = -\frac{i}{3} C_{ikj} - \frac{1}{2}(\eta_{ik}\Sigma_j + \eta_{kj}\Sigma_i) - i C_{ij}{}^m Q_{km}, \tag{3.12}$$

in which Q_{ij}'s are symmetric and traceless matrices with the following explicit forms:

$$Q_{11} = \frac{1}{3}\begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad Q_{22} = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad Q_{33} = \frac{-1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

$$Q_{21} = Q_{12} = \frac{1}{2}\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Q_{13} = Q_{31} = \frac{i}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$Q_{23} = Q_{32} = \frac{i}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{3.13}$$

The commutation relations (3.3) for the generators (3.9) follow from the relations (3.11). The pseudo-Hermiticity with respect to the Minkowskian metric η is also satisfied by Q_{ij}'s

$$Q_{ij}^\dagger = \eta Q_{ij} \eta^{-1}. \tag{3.14}$$

It is straightforward to show that the matrices Q_{ij} satisfy the following relations:

$$\begin{aligned} Q_{jm} Q_{ln} = {} & \frac{1}{6}\left(\eta_{jl}\eta_{mn} + \eta_{jn}\eta_{lm} - \frac{2}{3}\eta_{jm}\eta_{ln}\right) \\ & -\frac{1}{4}\left(\eta_{jl} Q_{mn} + \eta_{jn} Q_{lm} + \eta_{mn} Q_{jl} + \eta_{ml} Q_{jn} - \frac{4}{3}\eta_{jm} Q_{ln} - \frac{4}{3}\eta_{ln} Q_{jm}\right) \\ & -\frac{i}{8}\left(\eta_{jl} C_{mn}{}^p \Sigma_p + \eta_{jn} C_{ml}{}^p \Sigma_p + \eta_{ml} C_{jn}{}^p \Sigma_p + \eta_{mn} C_{jl}{}^p \Sigma_p\right). \end{aligned} \tag{3.15}$$

The identities (3.11), (3.12), and (3.15) are not pseudo unitary since, they are not closed with respect to the η-pseudo-Hermiticity. Indeed, their similar relations are also satisfied by $\Sigma_i^\dagger$'s and $Q_{ij}^\dagger$'s. It must be noted that the generators (3.9) for $su(1, 1)$ are related to Cartesian components of the spin 1 operator with the $su(2)$ commutation relations as (see, Ref. 31)

$$\Sigma_1 = -i S_x, \quad \Sigma_2 = -i S_y, \quad \Sigma_3 = S_z. \tag{3.16}$$

Furthermore, from comparison with[31] we have

$$\begin{aligned} & Q_{11} = \widehat{Q}_{xx}, && Q_{22} = \widehat{Q}_{yy}, && Q_{33} = -\widehat{Q}_{zz}, \\ & Q_{13} = Q_{31} = -i\widehat{Q}_{xz}, && Q_{12} = Q_{21} = \widehat{Q}_{xy}, && Q_{23} = Q_{32} = -i\widehat{Q}_{yz}, \end{aligned} \tag{3.17}$$

in which the matrices $\widehat{Q}_{ij}$ with $i, j = x, y, z$ are components of quadrupole tensor corresponding to the spin 1 operator with the $su(2)$ commutation relations.

Note that arbitrary operators u of the vector spaces $su(1, 1)$ corresponding to the spins $\frac{1}{2}$ and 1 are expressed as linear combinations of the traceless 2×2 and 3×3 matrices (3.6) and (3.9), respectively. Now we are in a position to derive the spin $\frac{1}{2}$ and 1 pseudo Dirac operators on the fuzzy AdS_2.

IV. GEOMETRY OF THE FUZZY AdS_2

Let us fix a positive integer l which controls the strength of noncommutativity, and promotes the coordinates x_i's of AdS_2 to play the role of the angular momentum generators in the nonunitary irreducible l-representation space. We denote the standard angular momentum operator on the commutative AdS_2 by $\mathbf{L}$. Besides, similar to Refs. 20, 32, and 33, let the fuzzy non-Hermitian matrix algebra be $\mathcal{A}_l = \{\alpha \in \mathbb{M}_{2l+1}(\mathbb{C})\}$. Every arbitrary element α can be expressed in terms of the bases, as a nonunitary l-representation of the $SU(1, 1)$, of the angular momentum $\mathbf{L} = (L_1, L_2, L_3)$:

$$[L_i, L_j] = iC_{ij}{}^k L_k. \tag{4.1}$$

The points on the fuzzy AdS_2 are distinguished from each other by the eigenvalues of the coordinate operators (L_1, L_2, L_3). However, the noncommutative geometry is a pointless geometry and this follows here from (4.1) that the operators L_1, L_2 and L_3 cannot be diagonalized simultaneously. To each angular momentum operator $\mathbf{L}$ we associate two linear operators $\mathbf{L}^L$ and $\mathbf{L}^R$ with the left- and right-actions on $\mathcal{A}_l$ $(i = 1, 2, 3)$

$$L_i^L\alpha = L_i\alpha, \qquad L_i^R\alpha = \alpha L_i, \qquad \forall\alpha \in \mathcal{A}_l. \tag{4.2}$$

We immediately conclude that left- and right- operators commute with each other

$$[L_i^L, L_j^R] = 0, \tag{4.3}$$

and consequently, $\mathcal{A}_l$ as a free bimodule carries the left- and right-regular representations of the fuzzy algebra

$$[L_i^L, L_j^L] = iC_{ij}{}^k L_k^L, \tag{4.4}$$

$$\mathbf{L}^L.\mathbf{L}^L = L_i^L \eta^{ij} L_j^L = l(1-l)1, \tag{4.5}$$

$$[L_i^R, L_j^R] = -iC_{ij}{}^k L_k^R, \tag{4.6}$$

$$\mathbf{L}^R.\mathbf{L}^R = L_i^R \eta^{ij} L_j^R = l(1-l)1. \tag{4.7}$$

In turn, (4.3) allows $\mathbf{L}^L$ and $\mathbf{L}^R$ to be used to define the pseudo fuzzy version of orbital momentum operator $\mathcal{L}_i := \mathrm{adL_i} = \mathrm{L_i^L} - \mathrm{L_i^R}$ on the fuzzy AdS_2. Indeed, $\mathcal{L}_i$ is defined by the adjoint action of L_i on the space $\mathcal{A}_l$: $\mathcal{L}_i\alpha := \mathrm{ad}_{\mathrm{L_i}}\alpha = [\mathrm{L_i}, \alpha]$. Then, (4.4) and (4.6) induce the commutation relations of the $su(1, 1)$ Lie algebra on the angular momentum operator

$$[\mathcal{L}_i, \mathcal{L}_j] = iC_{ij}{}^k \mathcal{L}_k. \tag{4.8}$$

The hypersurface AdS_2 with the negative constant curvature can be described by the coordinates x_i as a manifold embedded in the 3-dimensional flat Minkowskian space

$$\mathbf{x}.\mathbf{x} = x_i\eta^{ij}x_j = -1. \tag{4.9}$$

The commutative coordinates x_i as a limiting case are obtained from the fuzzy AdS_2

$$x_i = \lim_{l\to\infty} \frac{L_i^L}{l} = \lim_{l\to\infty} \frac{L_i^R}{l}. \tag{4.10}$$

This implies that the generators L_i^L and L_i^R do not tend to a finite limit at $l \to \infty$ whilst their difference is the angular momentum operator $\mathcal{L}_i$ in both of the fuzzy and commutative cases. Therefore, similar to what has been reported in Refs. 20, 32, and 33, l^{-1} plays the role of noncommutative parameter, so that one can obtain the orbital momentum operator on the commutative AdS_2 by the limiting process $l \to \infty$

$$\lim_{l\to\infty}(L_i^L - L_i^R) = -iC_{ij}{}^k x^j \frac{\partial}{\partial x^k}. \tag{4.11}$$

Although the operators L_i are not invariant under †, however, the representation of the $SU(1, 1)$ can be set as infinite-dimensional and unitary in the limit $l \to \infty$, and this means that the coordinates x_i given in (4.10) for the commutative AdS_2, are real-valued.

V. PSEUDO CHIRALITY AND DIRAC OPERATORS ON THE COMMUTATIVE AdS_2

Let γ be a Λ-pseudo $\mathbb{Z}_2$ grading (chirality) operator, $\gamma^2 = \mathrm{I}$ and $\gamma^\dagger = \Lambda\gamma\Lambda^{-1}$, which commutes with the total angular momentum $\mathcal{J}_i = \mathcal{L}_i + \Sigma_i$ and anticommutes with the Dirac operator: $[\gamma, \mathcal{J}_i] = 0$ and $\{\gamma, \mathcal{D}\} = 0$. $\mathcal{D}$, as an operator with first-order spatial derivatives, is defined as

$$\mathcal{D} = (\Sigma_i - \gamma\Sigma_i\gamma)\eta^{ij}(\mathcal{L}_j + \Sigma_j). \tag{5.1}$$

The finite-dimensional and nonunitary representation of $SU(1, 1)$ with the non-Hermitian generators Σ_1, Σ_2 and Σ_3 carries the spin degree of freedom for the Dirac field on AdS_2. The derivation of chirality operator γ is simple and will be discussed below.

A. Spin $\frac{1}{2}$ pseudo chirality and Dirac operators

In this case, the pseudo projector is proposed as a linear function of $\Sigma.\mathbf{x} = \Sigma_i\eta^{ij}x_j$: $P = a\Sigma.\mathbf{x} + b$ with a and b as appropriate constants. From $P^2 = P$ and (3.7), we find

$$P_\pm = \frac{1 \pm 2\Sigma.\mathbf{x}}{2}, \tag{5.2}$$

with the following properties:

$$P_\pm^2 = P_\pm, \qquad P_+ + P_- = 1, \qquad P_+P_- = P_-P_+ = 0, \qquad P_\pm^\dagger = \Sigma_3 P_\pm \Sigma_3^{-1}. \tag{5.3}$$

P_+ and P_- are two rank 1 pseudo projectors at each $\mathbf{x}$, i.e., $\mathrm{Tr}(P_\pm)$=1. Consequently, their corresponding chirality operators are

$$\gamma_\pm = 2P_\pm - 1 = \pm 2\Sigma.\mathbf{x}, \qquad \gamma_\pm^\dagger = \Sigma_3\gamma_\pm\Sigma_3^{-1}. \tag{5.4}$$

Thus, according to (5.1) we will have the same Dirac operator, up to a scaling factor as ± 1, for both γ_+ and γ_-:

$$\mathcal{D}_\pm = \pm(2\Sigma.\mathcal{L} - 1). \tag{5.5}$$

Furthermore, Eq. (3.8) implies that

$$\mathcal{D}^\dagger = \Sigma_3\mathcal{D}\Sigma_3^{-1}, \tag{5.6}$$

so, it is a Σ_3-pseudo Hermitian. The pseudo projectors $P_\pm$ at hand, we can introduce projective modules $P_\pm(\mathcal{A}_\mathbb{C})^2$ that carry the left- and right- $\mathcal{A}_\mathbb{C}$-actions as the same. From Eq. (5.3), it becomes obvious that the trivial module $(\mathcal{A}_\mathbb{C})^2$ is decomposed into a direct sum of two projective $\mathcal{A}_\mathbb{C}$-modules,

$$(\mathcal{A}_\mathbb{C})^2 = P_+(\mathcal{A}_\mathbb{C})^2 \oplus P_-(\mathcal{A}_\mathbb{C})^2\,. \tag{5.7}$$

B. Spin 1 pseudo chirality and Dirac operators

Pseudo projectors are suggested to be as quadratic expressions of $\Sigma.\mathbf{x}$: $P = a(\Sigma.\mathbf{x})^2 + b(\Sigma.\mathbf{x}) + c$. The constants a, b and c will be determined by imposing the condition that P satisfies the relation $P^2 = P$. On the other hand from (3.12) we see that

$$\begin{aligned}(\Sigma.\mathbf{x})^3 &= \Sigma_i \Sigma_m \Sigma_p \eta^{ij}\eta^{mn}\eta^{pq} x_j x_n x_q \\ &= \frac{-i}{3} C_{imp}\eta^{ij}\eta^{mn}\eta^{pq} x_j x_n x_q - \frac{1}{2}\Sigma_p \eta_{im}\eta^{ij}\eta^{mn}\eta^{pq} x_j x_n x_q \\ &\quad -\frac{1}{2}\Sigma_i \eta_{mp}\eta^{ij}\eta^{mn}\eta^{pq} x_j x_n x_q - iC_{ip}{}^{r} Q_{mr}\eta^{ij}\eta^{mn}\eta^{pq} x_j x_n x_q \\ &= -\frac{1}{2}\Sigma_p x^p \eta^{jn} x_j x_n - \frac{1}{2}\Sigma_i x^i \eta^{nq} x_n x_q \\ &= \Sigma.\mathbf{x},\end{aligned} \tag{5.8}$$

$$(\Sigma.\mathbf{x})^4 = (\Sigma.\mathbf{x})^2 = \Sigma_i \Sigma_j \eta^{ik}\eta^{jm} x_k x_m = \frac{2}{3} - Q_{ij} x^i x^j. \tag{5.9}$$

They lead to three different pseudo projectors with their corresponding chiralities

$$\begin{aligned} P_1 &= 1 - (\Sigma.\mathbf{x})^2, & \gamma_1 &= 1 - 2(\Sigma.\mathbf{x})^2, \\ P_2 &= \frac{(\Sigma.\mathbf{x})^2 - \Sigma.\mathbf{x}}{2}, & \gamma_2 &= (\Sigma.\mathbf{x})^2 - \Sigma.\mathbf{x} - 1, \\ P_3 &= \frac{(\Sigma.\mathbf{x})^2 + \Sigma.\mathbf{x}}{2}, & \gamma_3 &= (\Sigma.\mathbf{x})^2 + \Sigma.\mathbf{x} - 1, \end{aligned} \tag{5.10}$$

with $(i = 1, 2, 3)$

$$P_i^2 = P_i, \quad P_1 + P_2 + P_3 = 1, \quad P_1 P_2 = P_2 P_3 = P_3 P_1 = 0, \quad P_i^\dagger = \eta P_i \eta^{-1}, \tag{5.11}$$

and

$$\gamma_1 + \gamma_2 + \gamma_3 = -1, \qquad \gamma_i^\dagger = \eta \gamma_i \eta^{-1}. \tag{5.12}$$

Again at each $\mathbf{x}$, the pseudo projectors P_1, P_2 and P_3 are of rank 1. The Dirac operators corresponding to the η-pseudo Hermitian chirality operators γ_1, γ_2 and γ_3 can now be calculated using Eq. (5.1). For example, we first note that

$$\begin{aligned}\Sigma_i - \gamma_1 \Sigma_i \gamma_1 &= 2\Sigma_i(\Sigma.\mathbf{x})^2 + 2(\Sigma.\mathbf{x})^2\Sigma_i - 4(\Sigma.\mathbf{x})^2\Sigma_i(\Sigma.\mathbf{x})^2 \\ &= \left[\frac{-2i}{3}\left(C_{ikj} + C_{kji}\right) - \left(\eta_{ik}\Sigma_j + 2\eta_{kj}\Sigma_i + \eta_{ji}\Sigma_k\right) - 2iC_{ij}{}^{m} Q_{km} - 2iC_{ki}{}^{m} Q_{jm}\right] x^k x^j \\ &\quad -4\Sigma_m \Sigma_p \Sigma_i \eta^{mn}\eta^{pq} x_n x_q (\Sigma.\mathbf{x})^2 \\ &= -2(\Sigma.\mathbf{x})x_i + 2\Sigma_i - 4\Sigma_m \Sigma_p \Sigma_i \eta^{mn}\eta^{pq} x_n x_q (\Sigma.\mathbf{x})^2 \\ &= 2\Sigma_i - 2\Sigma_i(\Sigma.\mathbf{x})^2 + 4iC_{mi}{}^{r} Q_{pr} x^m x^p (\Sigma.\mathbf{x})^2 \\ &= \Sigma_i + (\Sigma.\mathbf{x})x_i + 2iC_{ip}{}^{r} Q_{mr} x^m x^p + 4iC_{mi}{}^{r} Q_{pr} x^m x^p (\Sigma.\mathbf{x})^2\end{aligned} \tag{5.13}$$

and

$$\begin{aligned}4iC_{mi}{}^{r} Q_{pr} x^m x^p (\Sigma.\mathbf{x})^2 &= -\frac{8i}{3} C_{mi}{}^{r} Q_{pr} x^m x^p + 4iC_{ki}{}^{m} Q_{jm} Q_{ln} x^k x^j x^l x^n \\ &= -2iC_{ki}{}^{m} Q_{mn} x^k x^n - \Sigma_i - (\Sigma.\mathbf{x})x_i,\end{aligned} \tag{5.14}$$

then

$$\Sigma_i - \gamma_1 \Sigma_i \gamma_1 = 2(\Sigma_i + (\Sigma.\mathbf{x})x_i). \tag{5.15}$$

Consequently, $\mathcal{D}_1$ is calculated as

$$\mathcal{D}_1 = 2(\Sigma.\mathcal{L} + (\Sigma.\mathbf{x})^2 - 2). \tag{5.16}$$

The coefficient 2 on the left hand side plays the role of a finite scaling coefficient. If we also consider the following equality:

$$\Sigma_i(\Sigma.\mathbf{x})\mathcal{L}^i = (\Sigma.\mathcal{L})(\Sigma.\mathbf{x}) - \Sigma.\mathbf{x}, \tag{5.17}$$

we can then derive $\mathcal{D}_2$ and $\mathcal{D}_3$ as

$$\mathcal{D}_2 = ((\Sigma.\mathbf{x})^2 + (\Sigma.\mathcal{L}) - 2) + 2\Sigma.\mathbf{x} - \{\Sigma.\mathbf{x}, \Sigma.\mathcal{L}\}, \tag{5.18}$$

$$\mathcal{D}_3 = ((\Sigma.\mathbf{x})^2 + (\Sigma.\mathcal{L}) - 2) - 2\Sigma.\mathbf{x} + \{\Sigma.\mathbf{x}, \Sigma.\mathcal{L}\}. \tag{5.19}$$

This immediately implies that $\mathcal{D}_1 = \mathcal{D}_2 + \mathcal{D}_3$. From (3.10) it follows that the three Dirac operators are pseudo Hermitian with respect to η, $(i = 1, 2, 3)$,

$$\mathcal{D}_i^\dagger = \eta\mathcal{D}_i\eta^{-1}. \tag{5.20}$$

Equation (5.11) present a type of splitting on the trivial module $(\mathcal{A}_\mathbb{C})^3$ in terms of three projective $\mathcal{A}_\mathbb{C}$-modules,

$$(\mathcal{A}_\mathbb{C})^3 = P_1\,(\mathcal{A}_\mathbb{C})^3 \oplus P_2\,(\mathcal{A}_\mathbb{C})^3 \oplus P_3\,(\mathcal{A}_\mathbb{C})^3\,. \tag{5.21}$$

VI. PSEUDO FUZZY CHIRALITY AND DIRAC OPERATORS

The pair of suitable generators Γ and Γ' of pseudo Ginsparg-Wilson algebra $\mathcal{A}$ can be used to construct two new Λ-pseudo Hermitian operators as

$$\begin{aligned} \Gamma_1 &= \frac{1}{2}\left(\Gamma + \Gamma'\right), \qquad & \Gamma_1^\dagger &= \Lambda\Gamma_1\Lambda^{-1}, \\ \Gamma_2 &= \frac{1}{2}\left(\Gamma - \Gamma'\right), \qquad & \Gamma_2^\dagger &= \Lambda\Gamma_2\Lambda^{-1}, \end{aligned} \tag{6.1}$$

so that, Γ_1 and Γ_2 anticommute with each other: $\{\Gamma_1, \Gamma_2\} = 0$. Based on the fuzzy model we suppose that one of them, having suitably scaled and taken the limit to infinity, tends to the Dirac operator and the other one to chirality operator on the commutative AdS_2. Nevertheless, we must emphasize that in the fuzzy version, the fundamental anticommutation relation between the chirality and Dirac operators is realized in the representation space of the total angular momentum containing both spin and orbital contributions. From the relations (3.3) and (4.8) it becomes obvious that the operators,

$$\mathcal{J}_i = \mathcal{L}_i + \Sigma_i, \tag{6.2}$$

as the components of total angular momentum on AdS_2 not only constitute the Lie algebra $su(1, 1)$

$$[\mathcal{J}_i, \mathcal{J}_j] = iC_{ij}{}^k\mathcal{J}_k, \tag{6.3}$$

but also satisfy the following commutation relations:

$$[\mathcal{L}_i, \mathcal{J}_j] = iC_{ij}{}^k\mathcal{L}_k, \qquad [\Sigma_i, \mathcal{J}_j] = iC_{ij}{}^k\Sigma_k. \tag{6.4}$$

l- and s-representations of $\mathbf{L}^L$ and Σ lead to the discrete $(l - s + k)$-representations with k as a non-negative integer number. Therefore, from coupling l- and s-representations we obtain representation spaces labeled by the values of k. Next we note that, k labels $(l - s + k)$-representation subspaces as V_k and they are distinguished from each other by the spectrum of $\Sigma.\mathbf{L}^L$

$$\mathrm{Spectrum of}\quad \Sigma.\mathbf{L}^\mathrm{L}\big|_{\mathrm{V_k}} = \mathrm{ls} + \frac{\mathrm{k}}{2}(1 - \mathrm{k} + 2\mathrm{s} - 2\mathrm{l}). \tag{6.5}$$

The spectrum for $\Sigma.\mathbf{L}^R$ will take its negative value. Λ-pseudo Hermitian operators Γ and Γ' are proposed to be chosen as $2P^L - 1$ and $2P^R - 1$, and depending on that P^L and P^R are projection operators on which one of the subspaces V_k, they will be idempotent over it.

A. Spin $\frac{1}{2}$ pseudo fuzzy chirality and Dirac operators

According to our discussions in Sec. III, for spin $\frac{1}{2}$ we have $\Lambda = \Sigma_3$. Then in this case, Γ and $\Gamma^{'}$ are Σ_3-pseudo Hermitian 2×2 matrices. Let us first explain how they can be constructed using the projectors. If we set the representation space to $V = V_0 \oplus V_1$, then we obtain the left projector operators as

$$P^L_{l-\frac{1}{2}} = \frac{2\Sigma.\mathbf{L}^L + l - 1}{2l - 1}, \qquad P^L_{l+\frac{1}{2}} = -\frac{2\Sigma.\mathbf{L}^L - l}{2l - 1}, \tag{6.6}$$

with the following values for their restriction to the subspaces V_0 and V_1:

$$P^L_{l-\frac{1}{2}}\Big|_{V_0} = P^L_{l+\frac{1}{2}}\Big|_{V_1} = 1, \qquad P^L_{l-\frac{1}{2}}\Big|_{V_1} = P^L_{l+\frac{1}{2}}\Big|_{V_0} = 0. \tag{6.7}$$

Note that V_0 and V_1 are the first two representation subspaces of the total angular momentum on fuzzy AdS_2. To construct the pseudo projectors (6.6) we have used the statements "spectrum of $\Sigma.\mathbf{L}^L\big|_{V_0,V_1} = \frac{l}{2}, \frac{1-l}{2}$." Therefore, $P^L_{l-\frac{1}{2}}$ and $P^L_{l+\frac{1}{2}}$ are the projector operators on the subspaces V_0 and V_1, respectively,

$$\left(P^L_{l-\frac{1}{2}}\right)^2\Big|_{V_0} = P^L_{l-\frac{1}{2}}\Big|_{V_0}, \qquad \left(P^L_{l+\frac{1}{2}}\right)^2\Big|_{V_1} = P^L_{l+\frac{1}{2}}\Big|_{V_1}. \tag{6.8}$$

The subscripts $l - \frac{1}{2}$ and $l + \frac{1}{2}$ are the labels corresponding to irreducible representation subspaces of the $su(1, 1)$ Lie algebra, i.e., $l - s + k$. In other words, $P^L_{l-\frac{1}{2}}$ and $P^L_{l+\frac{1}{2}}$ are the projector couplings l and $\frac{1}{2}$ to $l - \frac{1}{2}$ and $l + \frac{1}{2}$, respectively. Also, one can easily conclude that

$$P^L_{l-\frac{1}{2}} + P^L_{l+\frac{1}{2}} = 1, \qquad P^L_{l-\frac{1}{2}}P^L_{l+\frac{1}{2}}\Big|_{V_0} = P^L_{l-\frac{1}{2}}P^L_{l+\frac{1}{2}}\Big|_{V_1} = 0, \qquad P^{L\dagger}_{l\mp\frac{1}{2}} = \Sigma_3 P^L_{l\mp\frac{1}{2}}\Sigma_3^{-1}. \tag{6.9}$$

The symbol "†", i.e., the adjoint action, operates on the spin part only. The relations (6.9) induce the decomposition $(\mathcal{A}_l)^2 = P^L_{l-\frac{1}{2}}(\mathcal{A}_l)^2 \oplus P^L_{l+\frac{1}{2}}(\mathcal{A}_l)^2$ for the trivial module $(\mathcal{A}_l)^2$ in terms of the projective modules. Comparing Eq. (4.4) with (4.6), we find that the right projectors are the same as the left ones with $-\mathbf{L}^R$ replacing $\mathbf{L}^L$:

$$P^R_{l-\frac{1}{2}} = \frac{-2\Sigma.\mathbf{L}^R + l - 1}{2l - 1}, \qquad P^R_{l+\frac{1}{2}} = \frac{2\Sigma.\mathbf{L}^R + l}{2l - 1}. \tag{6.10}$$

This gives rise to the projection behavior quite similar to (6.7). Now, we can introduce two idempotent operators for either subspaces V_0 and V_1 as

$$\begin{aligned}
\Gamma^L_{l-\frac{1}{2}} &= 2P^L_{l-\frac{1}{2}} - 1 = \frac{4\Sigma.\mathbf{L}^L - 1}{2l - 1}, & \Gamma^R_{l-\frac{1}{2}} &= 2P^R_{l-\frac{1}{2}} - 1 = \frac{-4\Sigma.\mathbf{L}^R - 1}{2l - 1},\\
\Gamma^L_{l+\frac{1}{2}} &= 2P^L_{l+\frac{1}{2}} - 1 = \frac{-4\Sigma.\mathbf{L}^L + 1}{2l - 1}, & \Gamma^R_{l+\frac{1}{2}} &= 2P^R_{l+\frac{1}{2}} - 1 = \frac{4\Sigma.\mathbf{L}^R + 1}{2l - 1}.
\end{aligned} \tag{6.11}$$

The spin $\frac{1}{2}$ Σ_3-pseudo Hermitian fuzzy chirality and Dirac operators with the representation space V are

$$\begin{aligned}
\gamma^F_\pm &= \frac{1}{2}\left(\Gamma^L_{l\mp\frac{1}{2}} + \Gamma^R_{l\pm\frac{1}{2}}\right) = \pm\frac{2\Sigma.(\mathbf{L}^L + \mathbf{L}^R)}{2l - 1}, & \gamma^{F\dagger}_\pm &= \Sigma_3\gamma^F_\pm\Sigma_3^{-1},\\
\mathcal{D}^F_\pm &= l\left(\Gamma^L_{l\mp\frac{1}{2}} - \Gamma^R_{l\pm\frac{1}{2}}\right) = \pm l\frac{4\Sigma.\mathcal{L} - 2}{2l - 1}, & \mathcal{D}^{F\dagger}_\pm &= \Sigma_3\mathcal{D}^F_\pm\Sigma_3^{-1}.
\end{aligned} \tag{6.12}$$

Now, it is easy to check that the squares of fuzzy chirality operators tend to 1 in the limit $l \to \infty$,

$$\lim_{l\to\infty}\left(\gamma^F_\pm\right)^2 = 1, \tag{6.13}$$

and also their anticommutators with the fuzzy Dirac operators $\mathcal{D}^F_{\pm}$, vanish when they are restricted to V,

$$\left\{\mathcal{D}^F_{\pm}, \gamma^F_{\pm}\right\}\big|_V = 0. \tag{6.14}$$

Moreover, and most importantly, is the fact that the chirality and Dirac operators on commutative AdS_2, i.e., (5.4) and (5.5), are obtained as the limiting cases of the fuzzy versions $\gamma^F_{\pm}$ and $\mathcal{D}^F_{\pm}$,

$$\lim_{l\to\infty} \gamma^F_{\pm} = \gamma_{\pm}, \qquad \lim_{l\to\infty} \mathcal{D}^F_{\pm} = \mathcal{D}_{\pm}. \tag{6.15}$$

In particular, the left pseudo projectors of fuzzy version tend to the pseudo projectors of commutative case in the $l \to \infty$ limit,

$$\lim_{l\to\infty} P^L_{l-\frac{1}{2}} = P_+, \qquad \lim_{l\to\infty} P^L_{l+\frac{1}{2}} = P_-. \tag{6.16}$$

B. Spin 1 pseudo fuzzy chirality and Dirac operators

As explained in Sec. III, the interaction between spin 1 and orbital degrees of freedom on the fuzzy AdS_2 can be described by the η-pseudo Hermitian left and right angular momentum operators $\mathbf{L}^L$ and $\mathbf{L}^R$. Thus, η-pseudo Hermitian generators of the Ginsparg-Wilson algebra are 3×3 matrices with the entries in terms of $\mathbf{L}^L$ and $\mathbf{L}^R$. Consider $(l - 1)$-, l- and $(l + 1)$-representations V_0, V_1 and V_2 of the total angular momentum on fuzzy AdS_2, as the first three subspaces. Let us fix the representation space of the interaction term $\Sigma.\mathbf{L}^L$ as $V = V_0 \oplus V_1 \oplus V_2$ with "spectrum of $\Sigma.\mathbf{L}^L\big|_{V_0,V_1,V_2} = l, 1, 1 - l$", and likewise for $\mathbf{L}^R$. Therefore, the three 3×3 projector operators on the V_0, V_1 and V_2 can be constructed as

$$\begin{aligned}
P^L_{l-1} &= \frac{\left(\Sigma.\mathbf{L}^L - 1\right)\left(\Sigma.\mathbf{L}^L + l - 1\right)}{(l-1)(2l-1)},\\
P^L_{l} &= -\frac{\left(\Sigma.\mathbf{L}^L - l\right)\left(\Sigma.\mathbf{L}^L + l - 1\right)}{l(l-1)},\\
P^L_{l+1} &= \frac{\left(\Sigma.\mathbf{L}^L - l\right)\left(\Sigma.\mathbf{L}^L - 1\right)}{l(2l-1)}.
\end{aligned} \tag{6.17}$$

We have considered the projector coupling left angular momentum and spin operators to produce minimum total angular momenta $l - 1$, l and $l + 1$. The subspaces V_0, V_1 and V_2 are projected on themselves by the operators P^L_{l-1}, P^L_l and P^L_{l+1}, respectively,

$$\left(P^L_{l-1}\right)^2\Big|_{V_0} = P^L_{l-1}\big|_{V_0}, \qquad \left(P^L_{l}\right)^2\Big|_{V_1} = P^L_{l}\big|_{V_1}, \qquad \left(P^L_{l+1}\right)^2\Big|_{V_2} = P^L_{l+1}\big|_{V_2}. \tag{6.18}$$

In fact, Eq. (6.18) follow from their values when the pseudo projectors are restricted over the subspaces,

$$P^L_{l-1}\big|_{V_0} = P^L_{l}\big|_{V_1} = P^L_{l+1}\big|_{V_2} = 1, \qquad P^L_{l-1}\big|_{V_1,V_2} = P^L_{l}\big|_{V_0,V_2} = P^L_{l+1}\big|_{V_0,V_1} = 0. \tag{6.19}$$

Supplementary properties of the left projectors are

$$\begin{aligned}
&P^L_{l-1} + P^L_l + P^L_{l+1} = 1, \qquad P^{L\dagger}_{l\mp1} = \eta P^L_{l\mp1}\eta^{-1}, \qquad P^{L\dagger}_{l} = \eta P^L_{l}\eta^{-1},\\
&P^L_{l+1}P^L_l\big|_{V_0,V_1} = P^L_{l+1}P^L_l\big|_{V_0,V_2} = P^L_l P^L_{l-1}\big|_{V_0,V_2} = P^L_l P^L_{l-1}\big|_{V_1,V_2}\\
&\qquad = P^L_{l-1}P^L_{l+1}\big|_{V_1,V_2} = P^L_{l-1}P^L_{l+1}\big|_{V_0,V_1} = 0.
\end{aligned} \tag{6.20}$$

Next, we obtain a three-fold decomposition of the trivial module $(\mathcal{A}_l)^3$ in terms of the projective modules: $(\mathcal{A}_l)^3 = P^L_{l-1}(\mathcal{A}_l)^3 \oplus P^L_l(\mathcal{A}_l)^3 \oplus P^L_{l+1}(\mathcal{A}_l)^3$. The right projectors are again made by replacing $-\mathbf{L}^R$ for $\mathbf{L}^L$,

$$P^R_{l-1} = -\frac{\left(\Sigma.\mathbf{L}^R + 1\right)\left(-\Sigma.\mathbf{L}^R + l - 1\right)}{(l-1)(2l-1)},$$
$$P^R_{l} = \frac{\left(\Sigma.\mathbf{L}^R + l\right)\left(-\Sigma.\mathbf{L}^R + l - 1\right)}{l(l-1)}, \tag{6.21}$$
$$P^R_{l+1} = \frac{\left(\Sigma.\mathbf{L}^R + l\right)\left(\Sigma.\mathbf{L}^R + 1\right)}{l(2l-1)}.$$

Furthermore, three η-pseudo Hermitian generators of Ginsparg-Wilson algebra as idempotent operators on the subspaces V_0, V_1 and V_2, can be introduced as

$$\Gamma^L_{l-1} = 2P^L_{l-1} - 1 = \frac{2\left(\Sigma.\mathbf{L}^L - 1\right)\left(\Sigma.\mathbf{L}^L + l - 1\right) - (l-1)(2l-1)}{(l-1)(2l-1)},$$
$$\Gamma^L_{l} = 2P^L_{l} - 1 = \frac{-2\left(\Sigma.\mathbf{L}^L - l\right)\left(\Sigma.\mathbf{L}^L + l - 1\right) - l(l-1)}{l(l-1)}, \tag{6.22}$$
$$\Gamma^L_{l+1} = 2P^L_{l+1} - 1 = \frac{2\left(\Sigma.\mathbf{L}^L - l\right)\left(\Sigma.\mathbf{L}^L - 1\right) - l(2l-1)}{l(2l-1)}.$$

Also, the operators corresponding to the right action are obtained by substituting $-\mathbf{L}^R$ for $\mathbf{L}^L$. Now, we are in a position to introduce the spin 1 fuzzy chirality and Dirac operators on the representation space V,

$$\begin{aligned}
\gamma^F_1 &= \frac{1}{2}\left(\Gamma^L_l + \Gamma^R_l\right), & \mathcal{D}^F_1 &= l\left(\Gamma^L_l - \Gamma^R_l\right) + 2l\left(\Gamma^L_{l-1} - \Gamma^R_{l+1}\right) \\
& & &= l\left(\Gamma^R_l - \Gamma^L_l\right) + 2l\left(\Gamma^R_{l-1} - \Gamma^L_{l+1}\right), \\
\gamma^F_2 &= \frac{1}{2}\left(\Gamma^L_{l+1} + \Gamma^R_{l-1}\right), & \mathcal{D}^F_2 &= l\left(\Gamma^R_{l-1} - \Gamma^L_{l+1}\right), \\
\gamma^F_3 &= \frac{1}{2}\left(\Gamma^L_{l-1} + \Gamma^R_{l+1}\right), & \mathcal{D}^F_3 &= l\left(\Gamma^L_{l-1} - \Gamma^R_{l+1}\right).
\end{aligned} \tag{6.23}$$

It is clear that the spin 1 fuzzy chirality and Dirac operators are η-pseudo Hermitian, $(i = 1, 2, 3)$

$$\gamma^{F\dagger}_i = \eta\gamma^F_i\eta^{-1}, \qquad \mathcal{D}^{F\dagger}_i = \eta\mathcal{D}^F_i\eta^{-1}, \tag{6.24}$$

and also, chirality operators satisfy $\gamma^F_1 + \gamma^F_2 + \gamma^F_3 = -1$. Furthermore, one can easily show that the squares of the fuzzy chirality operators γ^F_1, γ^F_2 and γ^F_3 become 1 in the limit $l \to \infty$,

$$\lim_{l\to\infty}\left(\gamma^F_i\right)^2 = 1. \tag{6.25}$$

Furthermore, anticommutator of the fuzzy Dirac operators $\mathcal{D}^F_i$ and their corresponding chirality operators vanishes when restricted to V,

$$\left\{\mathcal{D}^F_i, \gamma^F_i\right\}\big|_V = 0. \tag{6.26}$$

In particular, chirality and Dirac operators given in (5.10), (5.16), (5.18), and (5.19) together with the pseudo projectors given in (5.10) for commutative AdS_2 can be derived as the limiting cases of their fuzzy versions,

$$\lim_{l\to\infty}\gamma^F_i = \gamma_i, \qquad \lim_{l\to\infty}\mathcal{D}^F_i = \mathcal{D}_i, \tag{6.27}$$

$$\lim_{l\to\infty}P^L_{l+1} = P_2, \qquad \lim_{l\to\infty}P^L_{l} = P_1, \qquad \lim_{l\to\infty}P^L_{l-1} = P_3. \tag{6.28}$$

To extract the results (6.27), calculations similar to the following will be helpful:

$$\begin{aligned}\lim_{l\to\infty}\frac{1}{2l}\left(\left(\Sigma.\mathbf{L}^L\right)^2-\left(\Sigma.\mathbf{L}^R\right)^2\right)&=\lim_{l\to\infty}\frac{1}{2l}\left(\left(\Sigma.\mathcal{L}+\Sigma.\mathbf{L}^R\right)^2-\left(\Sigma.\mathbf{L}^R\right)^2\right)\\&=\lim_{l\to\infty}\frac{1}{2l}\left(\Sigma.\mathcal{L}\right)^2+\lim_{l\to\infty}\frac{1}{2l}\left\{\Sigma.\mathcal{L},\,\Sigma.\mathbf{L}^R\right\}\\&=\frac{1}{2}\left\{\Sigma.\mathcal{L},\,\Sigma.\mathbf{x}\right\}.\end{aligned}\tag{6.29}$$

VII. CONCLUSION

In this paper, construction of the free and projective modules on the commutative and fuzzy AdS_2 is studied. The 2×2 and 3×3 nonunitary representations of $SU(1,1)$ are used to construct a pseudo generalization of the Ginsparg-Wilson algebra as well as the Dirac fields on commutative and fuzzy AdS_2. Based on the concept of pseudo-Hermiticity, we presented the pseudo chirality and Dirac operators and the projector coupling of the spins $\frac{1}{2}$ and 1 with angular momentum l on commutative AdS_2. Furthermore, the pseudo fuzzy operators with the spins $\frac{1}{2}$ and 1 are constructed by using the first two and three representation subspaces of the total angular momentum on the fuzzy AdS_2, respectively. It is also shown that the fuzzy versions of the pseudo chirality and Dirac operators and projectors are transformed to the commutative case in the limit $l\to\infty$.

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