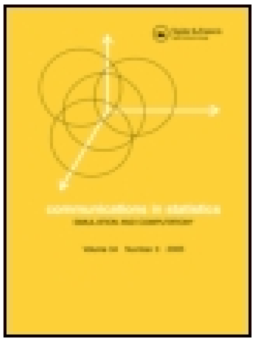

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Communications in Statistics - Simulation and Computation

ISSN: 0361-0918 (Print) 1532-4141 (Online) Journal homepage: http://www.tandfonline.com/loi/lssp20

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COMMUNICATIONS IN STATISTICS—SIMULATION AND COMPUTATION®
2017, VOL. 0, NO. 0, 1–21
https://doi.org/10.1080/03610918.2017.1332214

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Inference of stress-strength for the Type-II generalized logistic distribution under progressively Type-II censored samples

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ABSTRACT

This article studies the estimation of the reliability $R = P[Y < X]$ when X and Y come from two independent generalized logistic distributions of Type-II with different parameters, based on progressively Type-II censored samples. When the common scale parameter is unknown, the maximum likelihood estimator and its asymptotic distribution are proposed. The asymptotic distribution is used to construct an asymptotic confidence interval of R. Bayes estimator of R and the corresponding credible interval using the Gibbs sampling technique have been proposed too. Assuming that the common scale parameter is known, the maximum likelihood estimator, uniformly minimum variance unbiased estimator, Bayes estimation, and confidence interval of R are extracted. Monte Carlo simulations are performed to compare the different proposed methods. Analysis of a real dataset is given for illustrative purposes. Finally, methods are extended for proportional hazard rate models.

ARTICLE HISTORY
Received 23 November 2016
Accepted 12 May 2017

KEYWORDS
Bayes estimation; Credible interval; Maximum likelihood estimator; Monte Carlo simulations; Progressive Type-II censoring

MATHEMATICS SUBJECT CLASSIFICATION
62-07; 62F10; 62F25

1. Introduction

The random variable X has the Type-II generalized logistic (GL) distribution if it has the following cumulative distribution function (cdf):

$$F(x; \mu, \sigma, \alpha) = 1 - \left[\frac{e^{-(\frac{x-\mu}{\sigma})}}{1 + e^{-(\frac{x-\mu}{\sigma})}}\right]^{\alpha}, \quad -\infty < x < +\infty, \tag{1}$$

where $\mu \in \mathbb{R}$ and $\sigma, \alpha \in (0, +\infty)$. The probability density function (pdf) corresponding to the cdf (1) is

$$f(x; \mu, \sigma, \alpha) = \frac{\alpha e^{-\alpha(\frac{x-\mu}{\sigma})}}{\sigma(1 + e^{-\frac{x-\mu}{\sigma}})^{\alpha+1}}, \quad -\infty < x < +\infty. \tag{2}$$

Here μ, σ, and α are the location, scale, and shape parameters, respectively. In the particular case of $\alpha = 1$, F corresponds to the usual logistic distribution. If X is a random variable with Type-I GL distribution, then $-X$ has a Type-II GL distribution. Hence, Type-II model has properties similar to that of Type-I GL distribution. Several useful applications of the generalized logistic distribution have been discussed by George and Ojo (1980). The maximum

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likelihood estimator of (μ, σ, α) does not exist (see Zelterman, 1987). Therefore for convenience, without loss of generality, it is assumed $\mu = 0$.

The problem of interest in reliability analysis is inference of $R = P[X < Y]$. This problem arises naturally in the context of mechanical reliability of a system with stress X and strength Y. The system fails, if at any time the applied stress is greater than its strength. Also, stress-strength has applications in biometric, for example, if X presents the remaining lifetime of a patient treated by drug A and Y presents the remaining lifetime of another patient treated by drug B.

The term stress–strength in the reliability context was first applied by Church and Harris (1970). Various versions of this problem have been studied in literature. When X and Y are normally distributed, the maximum likelihood estimator (MLE) of $P[Y < X]$ has been considered by Downtown (1973). Awad et al. (1981) discussed the estimation of R, when X and Y have bivariate exponential distribution. Surles and Padgett (2001) considered the estimation of $P[Y < X]$, where X and Y are Burr Type-X random variable.

There are several works on the estimation of R based on the complete samples. For examples, Krishnamoorthy et al. (2007), Saraçoglu and Kaya (2007), Raqab et al. (2008), Karadayi et al. (2011), Saraçoglu et al. (2009), Kundu and Reqab (2009), Jiang and Wong (2008), Rezaei et al. (2010), Genç (2013), Huang and Wang (2012), and Babayi et al. (2014). While the estimation of R was discussed in the literature frequently for complete samples, but it has not received much interest for the Type-II progressive censoring samples case. In this context, the inferential procedures of R based on Type-II progressive censoring samples have been discussed by Asgharzadeh et al. (2011) when X and Y are two independent Weibull distributions with different scale parameters, but have the same shape parameter. The estimation of R for the exponential distribution under progressive censoring has been considered by Saraçoglu et al. (2012). As a new recently work, inference of stress–strength reliability for the exponential power (EP) distribution based on progressive Type-II censored samples has been studied by Akdam et al. (in press).

Among various censoring schemes, the Type-II progressive censoring scheme has become very popular one in the last decade. It can be described as follows: Let n units be on the life test at the same time. At the time of the first failure, r_1 of the $n - 1$ surviving units are chosen randomly and removed from the experiment. Similarly, at the time of the second failure r_2 of the $n - r_1 - 2$ surviving units are withdrawn and so on. Finally, at the time of the mth failure, all the remaining $n - r_1 - r_2 - \cdots - r_{m-1} - m$ surviving units are removed. Note that this scheme includes the conventional Type-II right censoring scheme, and it can be obtained by using $r_1 = r_2 = \cdots = r_{m-1} = 0$ and $r_m = n - m$. It is clear that complete sampling is achieved by using $r_1 = r_2 = \cdots = r_m = 0$. There are several advantages of the progressive censoring scheme than the usual Type-I and Type-II censoring schemes. For further details on progressively censoring and relevant references, see Balakrishnan and Aggarwala (2000) and Balakrishnan and Cramer (2014).

The main aim of this article is to discuss the inference of $P[Y < X]$, when X and Y have two independent Type-II generalized logistic (GL) distributions with different parameters based on Type-II progressive censoring samples. In Section 2, we drive the MLE of R using an iterative procedure when the common scale parameter is unknown. The asymptotic distribution of the MLE of R is given, and based on the asymptotic distribution of R the asymptotic confidence interval is also proposed. Bayes estimator of R and the corresponding credible interval using the Gibbs sampling technique have been proposed too, in this section. In Section 3, we consider different estimation of R when the common scale parameter is known. In this section, the MLE, the UMVUE, and Bayes estimation of R are discussed. The different proposed methods have been compared using Monte Carlo simulations and their results have been reported in Section 4. In Section 5, analysis of a real dataset is given for illustrative purposes. Finally, methods are extended for proportional hazard rate models in Section 6.

2. Maximum likelihood estimator of R with common scale parameter

In this section, we investigate the properties of R, when the scale parameter is σ for the two distributions, based on Type-II progressively censored samples.

Let $X \sim \mathrm{GL}(\alpha, \sigma)$ and $Y \sim \mathrm{GL}(\beta, \sigma)$, where X and Y are independent random variables. Therefore,

$$\begin{aligned} R = P(Y < X) &= \int_{-\infty}^{+\infty} \int_{-\infty}^{x} \frac{\alpha e^{-\alpha \frac{x}{\sigma}}}{\sigma(1+e^{-\frac{x}{\sigma}})^{\alpha+1}} \cdot \frac{\beta e^{-\beta \frac{y}{\sigma}}}{\sigma(1+e^{-\frac{y}{\sigma}})^{\beta+1}} dydx \\ &= 1 - \int_{-\infty}^{+\infty} \frac{\alpha e^{-(\alpha+\beta)\frac{x}{\sigma}}}{\sigma(1+e^{-\frac{x}{\sigma}})^{\alpha+\beta+1}} dx \\ &= \frac{\beta}{\alpha+\beta}. \end{aligned} \tag{3}$$

So, in order to obtain the MLE of R, we need to compute the MLE of α and β. Suppose $\mathbf{X} = (X_{1:m_1:n_1}, X_{2:m_1:n_1}, \ldots, X_{m_1:m_1:n_1})$ is a progressively Type-II censored sample from $\mathrm{GL}(\alpha, \sigma)$ with censored scheme $\mathbf{r} = (r_1, r_2, \ldots, r_{m_1})$ and $\mathbf{Y} = (Y_{1:m_2:n_2}, Y_{2:m_2:n_2}, \ldots, Y_{m_2:m_2:n_2})$ is also a progressively Type-II censored sample from $\mathrm{GL}(\beta, \sigma)$ with censored scheme $\mathbf{r}' = (r'_1, r'_2, \ldots, r'_{m_2})$. Therefore, the likelihood function of the observed sample is (see Balakrishnan and Aggarwala, 2000)

$$L(\alpha, \beta, \sigma) = \left[c_1 \prod_{i=1}^{m_1} f(x_{i:m_1:n_1})[1 - F(x_{i:m_1:n_1})]^{r_i} \right] \left[c_2 \prod_{i=1}^{m_2} f(y_{i:m_2:n_2})[1 - F(y_{i:m_2:n_2})]^{r'_i} \right]$$

where,

$$c_1 = n_1(n_1 - r_1 - 1) \cdots (n_1 - r_1 - r_2 - \cdots - r_{m_1-1} - m_1 + 1),$$

and

$$c_2 = n_2(n_2 - r'_1 - 1) \cdots (n_2 - r'_1 - r'_2 - \cdots - r'_{m_2-1} - m_2 + 1).$$

Therefore, the log-likelihood function of the observed sample is

$$\begin{aligned} l(\alpha, \beta, \sigma) = c &- (m_1 + m_2)\ln\sigma + m_1 \ln\alpha + m_2 \ln\beta - \frac{\alpha}{\sigma} \sum_{i=1}^{m_1} (r_i + 1) x_{i:m_1:n_1} \\ &- \sum_{i=1}^{m_1} (\alpha r_i + \alpha + 1) \ln(1 + e^{-\frac{x_{i:m_1:n_1}}{\sigma}}) - \frac{\beta}{\sigma} \sum_{i=1}^{m_2} (r'_i + 1) y_{i:m_2:n_2} \\ &- \sum_{i=1}^{m_2} (\beta r'_i + \beta + 1) \ln(1 + e^{-\frac{y_{i:m_2:n_2}}{\sigma}}). \end{aligned}$$

The MLEs of α, β, and σ say $\hat{\alpha}$, $\hat{\beta}$, and $\hat{\sigma}$, respectively, can be achieved as the solutions of

$$\frac{\partial l}{\partial \alpha} = \frac{m_1}{\alpha} - \sum_{i=1}^{m_1} (r_i + 1) \left[\frac{x_{i:m_1:n_1}}{\sigma} + \ln\left(1 + e^{-\frac{x_{i:m_1:n_1}}{\sigma}}\right) \right] = 0, \tag{4}$$

$$\frac{\partial l}{\partial \beta} = \frac{m_2}{\beta} - \sum_{i=1}^{m_2} (r_i' + 1)\left[\frac{y_{i:m_1:n_1}}{\sigma} + \ln\left(1 + e^{-\frac{y_{i:m_2:n_2}}{\sigma}}\right)\right] = 0, \tag{5}$$

$$\frac{\partial l}{\partial \sigma} = -\frac{m_1 + m_2}{\sigma} + \frac{\alpha}{\sigma^2}\sum_{i=1}^{m_1}(r_i + 1)x_{i:m_1:n_1} - \frac{1}{\sigma^2}\sum_{i=1}^{m_1}(\alpha r_i + \alpha + 1)\frac{x_{i:m_1:n_1} e^{-\frac{x_{i:m_1:n_1}}{\sigma}}}{1 + e^{-\frac{x_{i:m_1:n_1}}{\sigma}}}$$
$$+ \frac{\beta}{\sigma^2}\sum_{i=1}^{m_2}(r_i' + 1)y_{i:m_2:n_2} - \frac{1}{\sigma^2}\sum_{i=1}^{m_2}(\beta r_i' + \beta + 1)\frac{y_{i:m_2:n_2} e^{-\frac{y_{i:m_2:n_2}}{\sigma}}}{1 + e^{-\frac{y_{i:m_2:n_2}}{\sigma}}} = 0. \tag{6}$$

From (4), (5), and (6), we get

$$\hat{\alpha}_{(\hat{\sigma})} = \frac{-m_1}{\sum_{i=1}^{m_1}(r_i + 1)\ln\left(\frac{e^{-\frac{X_{i:m_1:n_1}}{\hat{\sigma}}}}{1+e^{-\frac{X_{i:m_1:n_1}}{\hat{\sigma}}}}\right)}, \tag{7}$$

$$\hat{\beta}_{(\hat{\sigma})} = \frac{-m_2}{\sum_{i=1}^{m_2}(r_i' + 1)\ln\left(\frac{e^{-\frac{Y_{i:m_2:n_2}}{\hat{\sigma}}}}{1+e^{-\frac{Y_{i:m_2:n_2}}{\hat{\sigma}}}}\right)}, \tag{8}$$

and $\hat{\sigma}$ can be given as the solution of the following nonlinear equation

$$h(\sigma) = \sigma, \tag{9}$$

where

$$h(\sigma) = (m_1 + m_2)^{-1}\left[\hat{\alpha}_{(\sigma)}\sum_{i=1}^{m_1}(r_i + 1)x_{i:m_1:n_1} - \sum_{i=1}^{m_1}(\hat{\alpha}_{(\sigma)} r_i + \hat{\alpha}_{(\sigma)} + 1)\frac{x_{i:m_1:n_1} e^{-\frac{x_{i:m_1:n_1}}{\sigma}}}{1 + e^{-\frac{x_{i:m_1:n_1}}{\sigma}}}\right.$$
$$\left. + \hat{\beta}_{(\sigma)}\sum_{i=1}^{m_2}(r_i' + 1)y_{i:m_2:n_2} - \sum_{i=1}^{m_2}(\hat{\beta}_{(\sigma)} r_i' + \hat{\beta}_{(\sigma)} + 1)\frac{y_{i:m_2:n_2} e^{-\frac{y_{i:m_2:n_2}}{\sigma}}}{1 + e^{-\frac{y_{i:m_2:n_2}}{\sigma}}}\right]. \tag{10}$$

Since $\hat{\sigma}$ is a fixed point solution of the nonlinear Eq. (10), therefore it can be achieved by using an iterative scheme as follows:

$$h(\sigma_{(j)}) = \sigma_{(j+1)}, \tag{11}$$

where $\sigma_{(j)}$ is the jth iterate of $\hat{\sigma}$. The iteration procedure should be stopped when $|\sigma_{(j)} - \sigma_{(j+1)}|$ is sufficiently small. First, $\hat{\sigma}$ is obtained, then $\hat{\alpha}$ and $\hat{\beta}$ can be resulted from (7) and (8), respectively. Since the ML estimators are invariant, so the MLE of R becomes

$$\hat{R} = \frac{\hat{\beta}_{(\hat{\sigma})}}{\hat{\alpha}_{(\hat{\sigma})} + \hat{\beta}_{(\hat{\sigma})}}. \tag{12}$$

2.1. Asymptotic distribution

In this section, first the asymptotic distribution of $\hat{\theta}' = (\hat{\alpha}, \hat{\beta}, \hat{\sigma})$ is considered, then the asymptotic distribution of $\hat{R}$ is extracted. Based on the asymptotic distribution of $\hat{R}$, the asymptotic confidence interval of R is managed. If $X_{1:m_1:n_1} < X_{2:m_1:n_1} < \cdots < X_{m_1:m_1:n_1}$ is a progressively Type-II censored sample from the GL(α, σ) distribution with censored scheme $\mathbf{r} = (r_1, r_2, \ldots, r_{m_1})$. Then $Z_{1:m_1:n_1} < Z_{2:m_1:n_1} < \cdots < Z_{m_1:m_1:n_1}$, where $Z_{i:m_1:n_1} = \frac{X_{i:m_1:n_1}}{\sigma}$ $(i = 1, \ldots, m_1)$ is a progressively Type-II censored sample from the standard Type-II GL distribution with censored scheme $\mathbf{r} = (r_1, r_2, \ldots, r_{m_1})$. We denote the Fisher information matrix of $\theta' = (\alpha, \beta, \sigma)$ as $I(\theta) = [I_{ij}(\theta)]$, $i, j = 1, 2, 3$. Therefore,

$$I(\theta) = -\begin{bmatrix} E\left(\frac{\partial^2 l}{\partial\alpha^2}\right) & E\left(\frac{\partial^2 l}{\partial\alpha\partial\beta}\right) & E\left(\frac{\partial^2 l}{\partial\alpha\partial\sigma}\right) \\ E\left(\frac{\partial^2 l}{\partial\beta\partial\alpha}\right) & E\left(\frac{\partial^2 l}{\partial\beta^2}\right) & E\left(\frac{\partial^2 l}{\partial\beta\partial\sigma}\right) \\ E\left(\frac{\partial^2 l}{\partial\sigma\partial\alpha}\right) & E\left(\frac{\partial^2 l}{\partial\sigma\partial\beta}\right) & E\left(\frac{\sigma^2 l}{\partial\sigma^2}\right) \end{bmatrix} = \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix},$$

where

$$\begin{aligned}
I_{11} &= -E\left(\frac{\partial^2 l}{\partial\alpha^2}\right) = \frac{m_1}{\alpha^2}, \\
I_{12} &= I_{21} = -E\left(\frac{\partial^2 l}{\partial\alpha\partial\beta}\right) = 0, \\
I_{22} &= -E\left(\frac{\partial^2 l}{\partial\beta^2}\right) = \frac{m_2}{\beta^2}, \\
I_{13} &= I_{31} = -E\left(\frac{\partial^2 l}{\partial\alpha\partial\sigma}\right) = \frac{1}{\sigma}\sum_{i=1}^{m_1}(r_i+1)E\left(\frac{-Z_{i:m_1:n_1}}{1+e^{-Z_{i:m_1:n_1}}}\right), \\
I_{23} &= I_{32} = -E\left(\frac{\partial^2 l}{\partial\beta\partial\sigma}\right) = \frac{1}{\sigma}\sum_{i=1}^{m_2}(r_i'+1)E\left(\frac{-W_{i:m_2:n_2}}{1+e^{-W_{i:m_2:n_2}}}\right), \\
I_{33} &= -E\left(\frac{\partial^2 l}{\partial\sigma^2}\right), \\
&= -\frac{m_1}{\sigma^2} + \frac{2\alpha}{\sigma^2}\sum_{i=1}^{m_1}(r_i+1)E(Z_{i:m_1:n_1}) - \frac{2}{\sigma^2}\sum_{i=1}^{m_1}(\alpha r_i+\alpha+1)E\left(\frac{Z_{i:m_1:n_1}e^{-Z_{i:m_1:n_1}}}{1+e^{-Z_{i:m_1:n_1}}}\right) \\
&\quad + \frac{1}{\sigma^2}\sum_{i=1}^{m_1}(\alpha r_i+\alpha+1)E\left(\frac{Z^2_{i:m_1:n_1}e^{-Z_{i:m_1:n_1}}}{(1+e^{-Z_{i:m_1:n_1}})^2}\right) \\
&\quad - \frac{m_2}{\sigma^2} + \frac{2\beta}{\sigma^2}\sum_{i=1}^{m_2}(r_i'+1)E(W_{i:m_2:n_2}) - \frac{2}{\sigma^2}\sum_{i=1}^{m_2}(\beta r_i'+\beta+1)E\left(\frac{W_{i:m_2:n_2}e^{-W_{i:m_2:n_2}}}{1+e^{-W_{i:m_2:n_2}}}\right) \\
&\quad + \frac{1}{\sigma^2}\sum_{i=1}^{m_2}(\beta r_i'+\beta+1)E\left(\frac{W^2_{i:m_2:n_2}e^{-W_{i:m_2:n_2}}}{(1+e^{-W_{i:m_2:n_2}})^2}\right),
\end{aligned}$$

where $W_{i:m_2:n_2} = \frac{Y_{i:m_2:n_2}}{\sigma}$.

Considering the results of Balakrishnan and Leung (1988), the pdf of $Z_{i:m_1:n_1}$ $(1 \le i \le m_1)$ is obtained as

$$f_{i:m_1:n_1}(z) = \frac{n_1!}{(i-1)!\,(n_1-i)!}\sum_{j=0}^{i-1}(-1)^j\binom{i-1}{j}\frac{\alpha e^{-\alpha(n_1+j-i+1)z}}{(1+e^{-z})^{\alpha(n_1+j-i+1)+1}}. \tag{13}$$

From (13) and some algebraic operations, we get

$$E\left(\frac{-Z_{i:m_1:n_1}}{1+e^{-Z_{i:m_1:n_1}}}\right) = \frac{n_1!}{(i-1)!\,(n_1-i)!}\sum_{j=0}^{i-1}(-1)^j\binom{i-1}{j}\frac{(\psi(\alpha b_{i,j})-\psi(2))}{b_{i,j}(\alpha b_{i,j}+1)},$$

$$E(Z_{i:m_1:n_1}) = \frac{n_1!}{(i-1)!\,(n_1-i)!}\sum_{j=0}^{i-1}(-1)^{j+1}\binom{i-1}{j}\frac{(\psi(\alpha b_{i,j})-\psi(1))}{b_{i,j}},$$

$$E\left(\frac{Z_{i:m_1:n_1}e^{-Z_{i:m_1:n_1}}}{1+e^{-Z_{i:m_1:n_1}}}\right) = \frac{n_1!}{(i-1)!\,(n_1-i)!}\sum_{j=0}^{i-1}(-1)^{j+1}\binom{i-1}{j}\frac{\alpha(\psi(\alpha b_{i,j}+1)-\psi(1))}{\alpha b_{i,j}+1},$$

$$E\left(\frac{Z^2_{i:m_1:n_1}\,e^{-Z_{i:m_1:n_1}}}{(1+e^{-Z_{i:m_1:n_1}})^2}\right) = \frac{n_1!}{(i-1)!\,(n_1-i)!}\sum_{j=0}^{i-1}(-1)^j\binom{i-1}{j}C(\alpha, b_{i,j}),$$

where

$$C(\alpha, b_{i,j}) = \frac{\alpha\{\psi'(\alpha b_{i,j}+1)+\psi'(2)+[\psi(\alpha b_{i,j}+1)-\psi(2)]^2\}}{(\alpha b_{i,j}+1)(\alpha b_{i,j}+2)},$$

and $b_{i,j} = n_1 + j - i + 1$, $\psi(x) = \frac{d\ln\Gamma(x)}{dx}$.

It is clear that results for $W_{i:m_2:n_2}$'s are similar therefore, we omit them here.

Under regularity conditions, the MLE of the vector $\theta' = (\alpha, \beta, \sigma)$ converges in distribution to normal distribution with the mean vector $\mu' = (\alpha, \beta, \sigma)$ and the covariance matrix $\Sigma = I^{-1}(\alpha, \beta, \sigma)$.

Theorem 2.1. *As $m_1 \to \infty$ and $m_2 \to \infty$ then*

$$[\sqrt{m_1}(\hat{\alpha}-\alpha), \sqrt{m_2}(\hat{\beta}-\beta), \sqrt{m_1}(\hat{\sigma}-\sigma)] \to N_3(0, U^{-1}(\alpha, \beta, \sigma)),$$

where

$$U(\alpha, \beta, \sigma) = \begin{bmatrix} u_{11} & 0 & u_{13} \\ 0 & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{bmatrix},$$

and

$$u_{11} = \frac{I_{11}}{m_1}, \quad u_{13} = u_{31} = \frac{I_{13}}{m_1}, \quad u_{22} = \frac{I_{22}}{m_2}, \quad u_{23} = u_{32} = \frac{I_{23}}{\sqrt{m_1 m_2}}, \quad u_{33} = \frac{I_{33}}{m_1}.$$

Proof. The proof follows from the asymptotic normality of MLEs under regularity conditions. □

Theorem 2.2. *As $m_1 \to \infty$ and $m_2 \to \infty$, then*

$$\sqrt{m_1}(\hat{R}-R) \to N(0, B),$$

where

$$B = \frac{1}{k(\alpha+\beta)^4}\left[\beta^2\left(u_{22}u_{33}-u_{23}^2\right) - 2\alpha\beta u_{23}u_{31} + \alpha^2\left(u_{11}u_{33}-u_{13}^2\right)\right],$$

and

$$k = u_{11}u_{22}u_{33} - u_{11}u_{23}u_{32} - u_{13}u_{22}u_{31}.$$

Proof. Here, $\hat{R} = g(\hat{\alpha}, \hat{\beta}, \hat{\sigma})$, where

$$g(\alpha, \beta, \sigma) = \frac{\beta}{\alpha + \beta},$$

Considering Theorem 2.1 and delta method, the proof is completed as follows:

$$B = b' U^{-1} b, \quad b = \begin{pmatrix} \frac{\partial g}{\partial \alpha} \\ \frac{\partial g}{\partial \beta} \\ \frac{\partial g}{\partial \sigma} \end{pmatrix} = \frac{1}{(\alpha+\beta)^2} \begin{pmatrix} -\beta \\ \alpha \\ 0 \end{pmatrix}.$$

□

Now using Theorem 2.2, asymptotic confidence interval of R is given by

$$\left(\hat{R} - z_{1-\gamma/2} \frac{\sqrt{\hat{B}}}{\sqrt{m_1}}, \hat{R} + z_{1-\gamma/2} \frac{\sqrt{\hat{B}}}{\sqrt{m_1}} \right).$$

To estimate variance B, the observed Fisher information matrix and the MLE of α, β, and σ is used.

2.2. Bayes estimation of R

In this subsection, we attempt to find the Bayes estimator of R under the assumption that the shape parameters α and β and the scale parameter σ are random variables. It is assumed that α, β, and σ have independent gamma priors with the parameters $\alpha \sim \text{Gamma}(a_1, b_1)$, $\beta \sim \text{Gamma}(a_2, b_2)$, and $\sigma \sim \text{Gamma}(a_3, b_3)$. Therefore,

$$\pi(\alpha) = \frac{b_1^{a_1}}{\Gamma(a_1)} \alpha^{a_1-1} e^{-b_1 \alpha}, \qquad \alpha > 0, \tag{14}$$

$$\pi(\beta) = \frac{b_2^{a_2}}{\Gamma(a_2)} \beta^{a_2-1} e^{-b_2 \beta}, \qquad \beta > 0, \tag{15}$$

and

$$\pi(\sigma) = \frac{b_3^{a_3}}{\Gamma(a_3)} \sigma^{a_3-1} e^{-b_3 \sigma}, \qquad \sigma > 0. \tag{16}$$

Here $a_1, b_1, a_2, b_2, a_3, b_3 > 0$.

Based on the above assumptions, the likelihood function of the observed data is

$$\begin{aligned} L(\text{data} \,|\alpha, \beta, \sigma) &= c_1 c_2 \sigma^{-(m_1+m_2)} \alpha^{m_1} \beta^{m_2} \\ &\times \exp\left\{ -\frac{\alpha}{\sigma} \sum_{i=1}^{m_1} (r_i + 1) x_{i:m_1:n_1} - \frac{\beta}{\sigma} \sum_{i=1}^{m_2} (r'_i + 1) y_{i:m_2:n_2} \right\} \\ &\times \prod_{i=1}^{m_1} \left(1 + e^{-\frac{x_{i:m_1:n_1}}{\sigma}}\right)^{-(\alpha r_i + \alpha + 1)} \prod_{i=1}^{m_2} \left(1 + e^{-\frac{y_{i:m_2:n_2}}{\sigma}}\right)^{-(\beta r'_i + \beta + 1)}. \end{aligned}$$

The joint density of the data, α, β, and σ can be achieved as

$$L(\text{data}, \alpha, \beta, \sigma) = L(\text{data} \,|\alpha, \beta, \sigma) \times \pi(\alpha) \times \pi(\beta) \times \pi(\sigma).$$

Therefore, the joint posterior density of α, β, and σ given the data is

$$L(\alpha, \beta, \sigma \,|\text{data}) = \frac{L(\text{data}, \alpha, \beta, \sigma)}{\int_0^\infty \int_0^\infty \int_0^\infty L(\text{data}, \alpha, \beta, \sigma) d\alpha d\beta d\sigma}. \tag{17}$$

Since (17) cannot be obtained analytically, we adopt the Gibbs sampling technique to compute the Bayes estimate of R and the corresponding credible interval of R. The posterior pdfs of α, β, and σ are as follows:

$$\alpha\,|\beta,\sigma,\text{data} \sim \text{Gamma}\left(a_1+m_1,\, b_1-\sum_{i=1}^{m_1}(r_i+1)\ln\left(\frac{e^{-\frac{x_{i:m_1:n_1}}{\sigma}}}{1+e^{-\frac{x_{i:m_1:n_1}}{\sigma}}}\right)\right),$$

$$\beta\,|\alpha,\sigma,\text{data} \sim \text{Gamma}\left(a_2+m_2,\, b_2-\sum_{i=1}^{m_2}(r_i+1)\ln\left(\frac{e^{-\frac{y_{i:m_2:n_2}}{\sigma}}}{1+e^{-\frac{y_{i:m_2:n_2}}{\sigma}}}\right)\right),$$

and

$$\pi(\sigma\,|\alpha,\beta,\text{data}) \propto \sigma^{a_3-m_1-m_2-1} \times \exp\left\{-b_3\sigma-\frac{\alpha}{\sigma}\sum_{i=1}^{m_1}(r_i+1)x_{i:m_1:n_1}-\frac{\beta}{\sigma}\sum_{i=1}^{m_2}(r_i'+1)y_{i:m_2:n_2}\right\} \times \prod_{i=1}^{m_1}\left(1+e^{-\frac{x_{i:m_1:n_1}}{\sigma}}\right)^{-(\alpha r_i+\alpha+1)}\prod_{i=1}^{m_2}\left(1+e^{-\frac{y_{i:m_2:n_2}}{\sigma}}\right)^{-(\beta r_i'+\beta+1)}. \quad (18)$$

The posterior pdf of σ is not known, Therefore, the algorithm of Gibbs sampling is used as follows:

Step 1: Start with an initial guess $(\alpha^{(0)},\beta^{(0)},\sigma^{(0)})$.
Step 2: Set $t=1$.
Step 3: Generate $\alpha^{(t)}$ from $\text{Gamma}(a_1+m_1, b_1-\sum_{i=1}^{m_1}(r_i+1)\ln(\frac{e^{-\frac{x_{i:m_1:n_1}}{\sigma^{(t-1)}}}}{1+e^{-\frac{x_{i:m_1:n_1}}{\sigma^{(t-1)}}}}))$.
Step 4: Generate $\beta^{(t)}$ from $\text{Gamma}(a_2+m_2, b_2-\sum_{i=1}^{m_2}(r_i+1)\ln(\frac{e^{-\frac{y_{i:m_2:n_2}}{\sigma^{(t-1)}}}}{1+e^{-\frac{y_{i:m_2:n_2}}{\sigma^{(t-1)}}}}))$.
Step 5: Using Metropolis–Hastings, generate $\sigma^{(t)}$ from $\pi(\sigma|\alpha^{(t-1)},\beta^{(t-1)},\text{data})$ with the $N(\sigma^{(t-1)},1)$ as a proposal distribution.
Step 6: Compute $R^{(t)}$ from (3).
Step 7: Set $t=t+1$.
Step 8: Repeat steps 3–7, T times.

Now the approximate posterior mean, and posterior variance of R become

$$\hat{E}(R\,|\text{data}) = \frac{1}{T-K}\sum_{t=K+1}^{T}R^{(t)}, \quad (19)$$

and

$$\hat{V}(R\,|\text{data}) = \frac{1}{T-K}\sum_{t=K+1}^{T}(R^{(t)}-\hat{E}(R\,|\text{data}))^2,$$

where K is the burn-in period.

Based on T and R values, using the method proposed by Chen and Shao (1999), the approximate highest posterior density (HPD) credible interval of R can be easily constructed. Let $R_{(K+1)} < R_{(K+2)} < \cdots < R_{(T-K)}$ be the ordered $R^{(t)}$, and suppose we would like to construct a $100(1-\gamma)\%$ approximate HPD credible interval of R, then consider the following:

$$\{(R_{(T-K)}, R_{((1-\gamma)(T-K))}),\ldots,(R_{(\gamma(T-K))}, R_{(T-K)})\}.$$

Choose that interval that has the shortest length.

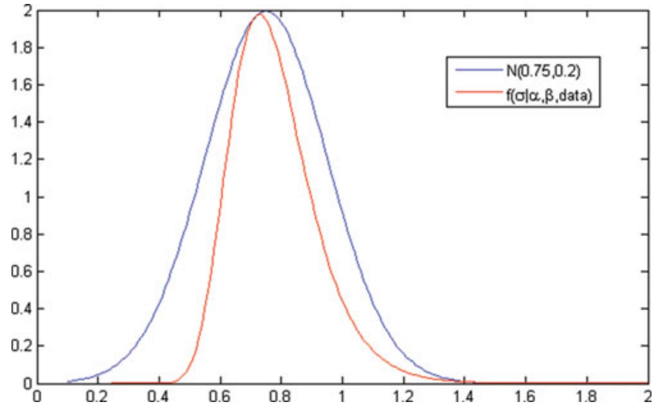

Figure 1. Posterior density function of scale parameter and normal distribution.

Note that in Step 5, we plot pdf of $f(\sigma|\alpha, \beta, \text{data})$ to find a suitable proposal density. Figure 1 shows that it is similar to normal distribution. So, to generate random number from $f(\sigma|\alpha, \beta, \text{data})$, we use the normal distribution for our proposal density. Actually, the normal distribution is a proposal distribution, which is located at the top of the target density and by the proposal distribution, we generate random number from $f(\sigma|\alpha, \beta, \text{data})$. Figure 2 presents plot of Metropolis Markov chain normal distribution as the proposal distribution.

3. Estimation of R if σ is known

In this section, the estimation of R when σ is known, is considered. Without loss of generality, we assume that $\sigma = 1$.

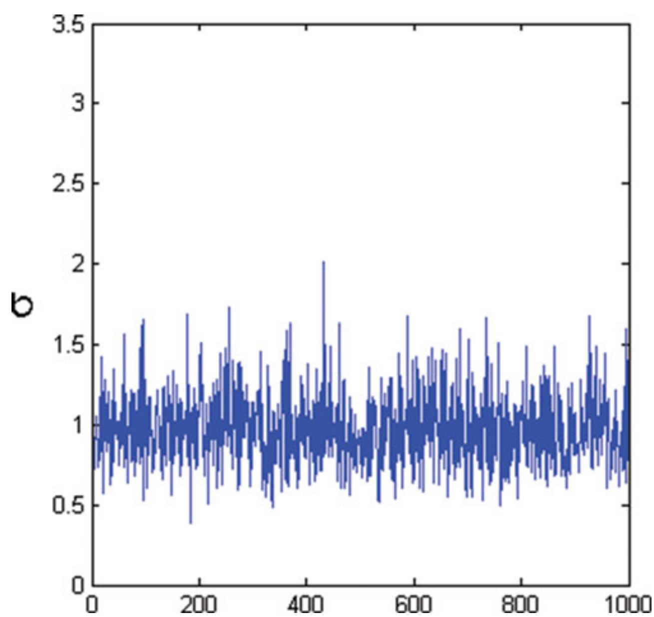

Figure 2. Metropolis Markov chain normal distribution as the proposal distribution.

3.1. MLE of R

Based on Section 3, it is clear that the MLE of R will be

$$\hat{R} = \frac{\hat{\beta}}{\hat{\alpha} + \hat{\beta}} = \frac{1}{1 + \frac{m_1 T_2}{m_2 T_1}}, \tag{20}$$

where $T_1 = -\sum_{i=1}^{m_1} (r_i + 1)\ln(\frac{e^{-X_{i:m_1:n_1}}}{1+e^{-X_{i:m_1:n_1}}})$ and $T_2 = -\sum_{i=1}^{m_2} (r_i' + 1)\ln(\frac{e^{-Y_{i:m_2:n_2}}}{1+e^{-Y_{i:m_2:n_2}}})$. Let

$$Z_1 = -n_1 \ln\left(\frac{e^{-X_{1:m_1:n_1}}}{1 + e^{-X_{1:m_1:n_1}}}\right), \tag{21}$$

and

$$Z_i = \left(n_1 - \sum_{k=1}^{i-1} r_k - i + 1\right)\left[-\ln\left(\frac{e^{-X_{i:m_1:n_1}}}{1 + e^{-X_{i:m_1:n_1}}}\right) - \left(-\ln\left(\frac{e^{-X_{i-1:m_1:n_1}}}{1 + e^{-X_{i-1:m_1:n_1}}}\right)\right)\right],$$
$$i = 1, 2, \ldots, m_1. \tag{22}$$

Balakrishnan and Aggarwala (2000) proved that Z_i's in (21) and (22) are independent and identically distributed (i.i.d) exponential random variables with mean $\frac{1}{\alpha}$. Therefore,

$$T_1 = -\sum_{i=1}^{m_1} (r_i + 1)\ln\left(\frac{e^{-X_{i:m_1:n_1}}}{1 + e^{-X_{i:m_1:n_1}}}\right) = \sum_{i=1}^{m_1} Z_i \sim \text{Gamma}(m_1, \alpha). \tag{23}$$

From (23) it is clear that $2\alpha T_1$ has the chi-square distribution with $2m_1$ degrees of freedom and similarly $2\beta T_2$ has the chi-square distribution with $2m_2$ degrees of freedom. Therefore,

$$\hat{R} \sim \frac{1}{1 + \frac{\alpha}{\beta} F} \quad \text{or} \quad \frac{R}{1 - R} \times \frac{1 - \hat{R}}{\hat{R}} \sim F, \tag{24}$$

where the random variable F has an $F_{2m_2,2m_1}$ distribution with $2m_2$ and $2m_1$ degrees of freedom. Therefore, the pdf of $\hat{R}$ is as follows:

$$f_{\hat{R}}(r) = \frac{1}{B(m_2, m_1) r^2}\left(\frac{m_2\beta}{m_1\alpha}\right)^{m_2} \frac{(\frac{1-r}{r})^{m_2-1}}{\left(1 + \frac{m_2\beta}{m_1\alpha}\left(\frac{1-r}{r}\right)\right)^{m_1+m_2}}, \quad 0 < r < 1. \tag{25}$$

The $100(1-\gamma)\%$ confidence interval of R can be obtained as

$$\left[\frac{1}{1 + F_{2m_1,2m_2;1-\gamma/2}(\frac{1}{\hat{R}} - 1)}, \frac{1}{1 + F_{2m_1,2m_2;\gamma/2}(\frac{1}{\hat{R}} - 1)}\right], \tag{26}$$

where $F_{2m_1,2m_1;\gamma/2}$ and $F_{2m_1,2m_2;1-\gamma/2}$ are the lower and upper $\frac{\gamma}{2}$th percentile points of an F distribution with $2m_1$ and $2m_2$ degrees of freedom.

3.2. UMVUE of R

Here, we obtain the UMVUE of R based on $\mathbf{X} = (X_{1:m_1:n_1}, X_{2:m_1:n_1}, \ldots, X_{m_1:m_1:n_1})$ and $\mathbf{Y} = (Y_{1:m_2:n_2}, Y_{2:m_2:n_2}, \ldots, Y_{m_2:m_2:n_2})$ when the common scale parameter σ is known. To do this, we recall the following Lemma, which will be used to in extracting our results. The proof is easy, so we omit it.

Lemma 3.1. *Let $X_1, X_2, \ldots, X_n$ be i.i.d random variables from exponential distribution with mean $\frac{1}{\lambda}$. Then, the condition density function of X_1 given $Y = \sum_{i=1}^{n} X_i$ is*

$$f_{X_1|Y}(x_1|y) = \frac{n-1}{y}\left(1-\frac{x_1}{y}\right)^{n-2}, \quad 0 < x_1 < y. \tag{27}$$

Theorem 3.1. *Based a jointly complete sufficient statistic (U, V) for (α, β), the UMVUE of R is*

$$\tilde{R} = \begin{cases} 1 - \sum_{k=0}^{m_2-1}(-1)^k \frac{\binom{m_2-1}{k}}{\binom{m_1+k-1}{k}}\left(\frac{U}{V}\right)^k & \text{if} \quad U \le V \\ \sum_{k=0}^{m_1-1}(-1)^k \frac{\binom{m_1-1}{k}}{\binom{m_2+k-1}{k}}\left(\frac{V}{U}\right)^k & \text{if} \quad V \le U \end{cases} \tag{28}$$

where $U = -\sum_{i=1}^{m_1}(r_i+1)\ln\left(\frac{e^{-X_{i:m_1:n_1}}}{1+e^{-X_{i:m_1:n_1}}}\right)$ *and* $V = -\sum_{i=1}^{m_2}(r_i'+1)\ln\left(\frac{e^{-Y_{i:m_2:n_2}}}{1+e^{-Y_{i:m_2:n_2}}}\right)$.

Proof. It is obvious that $-n_1\ln\left(\frac{e^{-X_{1:m_1:n_1}}}{1+e^{-X_{1:m_1:n_1}}}\right)$ and $-n_2\ln\left(\frac{e^{-Y_{1:m_2:n_2}}}{1+e^{-Y_{1:m_2:n_2}}}\right)$ have exponential distribution with $\frac{1}{\alpha}$ and $\frac{1}{\beta}$, respectively. Let us define

$$\phi(X_{1:m_1:n_1}, Y_{1:m_2:n_2}) = \begin{cases} 1 \text{ if} & -n_2\ln\left(\frac{e^{-Y_{1:m_2:n_2}}}{1+e^{-Y_{1:m_2:n_2}}}\right) < -n_1\ln\left(\frac{e^{-X_{1:m_1:n_1}}}{1+e^{-X_{1:m_1:n_1}}}\right) \\ 0 \text{ if} & -n_2\ln\left(\frac{e^{-Y_{1:m_2:n_2}}}{1+e^{-Y_{1:m_2:n_2}}}\right) > -n_1\ln\left(\frac{e^{-X_{1:m_1:n_1}}}{1+e^{-X_{1:m_1:n_1}}}\right). \end{cases} \tag{29}$$

Clearly, $\phi(X_{1:m_1:n_1}, Y_{1:m_2:n_2})$ is an unbiased estimator of R. Hence, by using this fact that (U, V) is a jointly complete sufficient statistic and applying Lehmann and Scheffe Theorem (Lehmann and Scheffe, 1950), the UMVUE of R is given by

$$\begin{aligned} \tilde{R} &= E(\phi(X_{1:m_1:n_1}, Y_{1:m_2:n_2})|U=u, V=v) \\ &= P\left(-n_2\ln\left(\frac{e^{-Y_{1:m_2:n_2}}}{1+e^{-Y_{1:m_2:n_2}}}\right) < -n_1\ln\left(\frac{e^{-X_{1:m_1:n_1}}}{1+e^{-X_{1:m_1:n_1}}}\right)|U=u, V=v\right) \\ &= \int\int_B f_{Z_1|U=u}(z_1|u) f_{W_1|V=v}(w_1|v)\,dz_1 dw_1, \end{aligned} \tag{30}$$

where $B = \{(z_1, w_1)|0 < z_1 < \frac{u}{n_1}, 0 < w_1 < \frac{v}{n_2}, n_1 z_1 > n_2 w_1\}$ and $Z_1 = -\ln\left(\frac{e^{-X_{1:m_1:n_1}}}{1+e^{-X_{1:m_1:n_1}}}\right)$ and $W_1 = -\ln\left(\frac{e^{-Y_{1:m_2:n_2}}}{1+e^{-Y_{1:m_2:n_2}}}\right)$. By using Lemma (3.1) for $u < v$, the right-hand side Eq. (30) can be expressed as

$$\begin{aligned} \tilde{R} &= \int_0^{u/n_1}\int_0^{n_1 z_1/n_2} n_1(m_1-1)\frac{(u-n_1z_1)^{m_1-2}}{u^{m_1-1}} n_2(m_2-1)\frac{(v-n_2w_1)^{m_2-2}}{v^{m_2-1}}\,dw_1 dz_1 \\ &= 1 - \frac{n_1(m_1-1)}{u^{m_1-1}v^{m_2-1}}\int_0^{u/n_1}(u-n_1z_1)^{m_1-2}(v-n_1z_1)^{m_2-1}dz_1 \\ &= 1-(m_1-1)\int_0^1 (1-t)^{m_1-2}\left(1-\frac{ut}{v}\right)^{m_2-1}dt. \end{aligned} \tag{31}$$

Considering the fact that

$$\left(1-\frac{ut}{v}\right)^{m_2-1} = \sum_{k=0}^{m_2-1}(-1)^k\binom{m_2-1}{k}\left(\frac{ut}{v}\right)^k,$$

then, $\tilde{R}$ for $u < v$ is simplified as

$$\tilde{R} = 1 - \sum_{k=0}^{m_2-1} (-1)^k \frac{\binom{m_2-1}{k}}{\binom{m_1+k-1}{k}} \left(\frac{u}{v}\right)^k.$$

Similarly, for $v < u$ we obtain

$$\tilde{R} = \sum_{k=0}^{m_1-1} (-1)^k \frac{\binom{m_1-1}{k}}{\binom{m_2+k-1}{k}} \left(\frac{v}{u}\right)^k.$$

□

3.3. Bayes estimator of R

In this subsection, we derive the Bayes estimator of R under the assumption that the shape parameters α and β are random variables. It is assumed that α and β have independent gamma priors with the pdfs (14) and (15), respectively. Therefore, the posterior pdfs of α and β are as follows:

$$\alpha|\text{data} \sim \text{Gamma}(a_1 + m_1, b_1 + T_1), \tag{32}$$

$$\beta|\text{data} \sim \text{Gamma}(a_2 + m_2, b_2 + T_2), \tag{33}$$

where $T_1 = -\sum_{i=1}^{m_1} (r_i + 1) \ln(\frac{e^{-X_{i:m_1:n_1}}}{1+e^{-X_{i:m_1:n_1}}})$ and $T_2 = -\sum_{i=1}^{m_2} (r'_i + 1) \ln(\frac{e^{-Y_{i:m_2:n_2}}}{1+e^{-Y_{i:m_2:n_2}}})$. Since the prior distributions α and β are independent, using (32) and (33) the posterior pdf of R becomes

$$f_R(r) = C \frac{r^{a_2+m_2-1}(1-r)^{a_1+m_1-1}}{((b_1+T_1)(1-r) + (b_2+T_2)r)^{m_2+m_1+a_1+a_2}}, \qquad 0 < r < 1,$$

where

$$C = \frac{\Gamma(m_2 + m_1 + a_1 + a_2)}{\Gamma(m_1 + a_1)\Gamma(m_2 + a_2)} (b_1 + T_1)^{a_1+m_1} (b_2 + T_2)^{a_2+m_2}.$$

It is not possible to take the explicit expressions for the posterior mean or median. On the other hand, the posterior mode can be easily achieved

$$\frac{d}{dr} f_R(r) = \frac{r^{A_2-1}(1-r)^{A_1-1}[2r^2(B_2 - B_1) + r(2B_1 - 2B_2 - A_1B_2 - A_2B_1) + A_2B_1]}{[B_1(1-r) + B_2 r]^{A_1+A_2+3}},$$

where $B_1 = b_1 + T_1,\ B_2 = b_2 + T_2,\ A_1 = a_1 + m_1 - 1$, and $A_2 = a_2 + m_2 - 1$.

It is clear that, for $r \in (0, 1)$, the $\frac{d}{dr} f_R(r) = 0$ has only two roots. Using the fact that $\lim_{r\to 0^+} \frac{d}{dr} f_R(r) > 0$ and $\lim_{r\to 0^-} \frac{d}{dr} f_R(r) < 0$, it easily follows that the density function $f_R(r)$ has a unique mode. The posterior mode can be achieved as the unique root that lies between 0 and 1 of the following quadratic equation:

$$2r^2(B_1 - B_2) + r(2B_2 - 2B_1 + A_1B_2 + A_2B_1) - A_2B = 0.$$

Now, consider the following loss function:

$$L(a, b) = \begin{cases} 1 & \text{if} \quad |a - b| > c \\ 0 & \text{if} \quad |a - b| \le c. \end{cases} \tag{34}$$

Note that the Bayes estimate with respect to the above equation is the midpoint of the "modal interval" of length $2c$ of the posterior distribution (see Ferguson, 1967). Therefore, the posterior mode is an approximate Bayes estimator of R with respect to the loss function (34), when the constant c is small.

As we mentioned before, the Bayes estimate of R under the squared error loss function cannot be obtained analytically. Alternatively, via the approximation of Lindley (1980) and following the approach of Ahmad et al. (1997), it can be seen that the approximate Bayes estimate of R, say $\hat{R}_{\text{Bayes}}$ under the squared error loss function is

$$\hat{R}_{\text{Bayes}} = \tilde{R}\left[1 + \frac{\tilde{\alpha}\tilde{R}^2(\tilde{\alpha}(m_1 + a_1 - 1) - \tilde{\beta}(m_2 + a_2 - 1))}{\tilde{\beta}^2(m_1 + b_1 - 1)(m_2 + a_2 - 1)}\right], \tag{35}$$

where

$$\tilde{R} = \frac{\tilde{\beta}}{\tilde{\alpha} + \tilde{\beta}}, \quad \tilde{\alpha} = \frac{m_1 + a_1 - 1}{b_1 + T_1}, \quad \text{and } \tilde{\beta} = \frac{m_2 + a_2 - 1}{b_2 + T_2}.$$

4. Simulation results

In this section, we mainly present some Mont Carlo simulation results in order to observe the behavior of the performance of the different methods for different censoring schemes, and for different parameter values. Two cases separately are considered to draw inference on R, namely, when (i) σ is unknown, (ii) σ is known.

In the first case, we compare the performances of the MLE and the Bayes estimates (with respect to the squared error loss function) in terms of biases, and mean squares errors (MSE). We also compare two confidence intervals, namely, the confidence intervals obtained by using asymptotic distributions of the MLE and the HPD credible intervals in terms of the average confidence lengths, and coverage percentages.

We apply different parameter values, different hyper parameters, and different sampling schemes. We used three sets of parameter values $(\alpha = 1, \beta = 1.5, \sigma = 0.5)$, $(\alpha = 1, \beta = 0.5, \sigma = 1)$, and $(\alpha = 2, \beta = 2.5, \sigma = 1.5)$ to compare the MLE and Bayes estimator. With using the algorithm of Gibbs sampling in Section 2 for computing the Bayes estimators and HPD credible intervals, we determine $T = 2{,}000$ and $K = 1{,}000$ experimentally so that the results of biases and variances will not be affected by the initial values of the parameters. For computing the Bayes estimators and HPD credible intervals, we assume three priors as follows:

$$\begin{aligned}
&\textbf{Prior 1}: \quad a_j = 0.0001, \quad b_j = 0.0001, \quad j = 1, 2, 3,\\
&\textbf{Prior 2}: \quad a_j = 1, \quad b_j = 1, \quad j = 1, 2, 3,\\
&\textbf{Prior 3}: \quad a_j = 2, \quad b_j = 3, \quad j = 1, 2, 3.
\end{aligned}$$

We also consider three censoring schemes as given in Table 1. For different parameter values, different censoring schemes, and different priors, we report the average biases, and MSE of the

Table 1. Censoring schemes.

Scheme	(m, n)	C.S.
r_1	(10, 30)	(0, 0, 0, 0, 0, 0, 0, 0, 0, 20)
r_2	(10, 30)	(20, 0, 0, 0, 0, 0, 0, 0, 0, 0)
r_3	(10, 30)	(2, 2, 2, 2, 2, 2, 2, 2, 2, 2)

Table 2. Biases and MSE of the MLE and Bayes estimators of R when σ is unknown.

(α, β, σ)	C.S.		MLE	Bayes prior 1	Bayes prior 2	Bayes prior 3
(1, 1.5, 0.5)	(r_1, r_1)	Bias	− 0.0512	− 0.0621	− 0.0607	− 0.0548
		MSE	0.0181	0.0217	0.0226	0.0202
	(r_1, r_2)	Bias	0.0443	− 0.0603	0.0539	− 0.0524
		MSE	0.0123	0.0175	0.0167	0.0150
	(r_1, r_3)	Bias	− 0.0334	− 0.0482	− 0.0432	0.0397
		MSE	0.0213	0.0273	0.0264	0.0236
	(r_2, r_2)	Bias	0.0153	0.0244	0.0237	0.0217
		MSE	0.0115	0.0167	0.0162	0.0151
	(r_2, r_3)	Bias	0.0133	0.0215	− 0.0196	− 0.0188
		MSE	0.0135	0.0177	0.0169	0.0162
	(r_3, r_3)	Bias	− 0.0314	− 0.0476	− 0.0439	− 0.0380
		MSE	0.0126	0.0180	0.0179	0.0153
(1, 0.5, 1)	(r_1, r_1)	Bias	− 0.0463	− 0.0604	0.0576	0.0531
		MSE	0.0215	0.0252	0.0247	0.0236
	(r_1, r_2)	Bias	− 0.0343	0.0501	− 0.0477	− 0.0458
		MSE	0.0173	0.0239	0.0220	0.0211
	(r_1, r_3)	Bias	− 0.0273	− 0.0340	0.0314	− 0.0301
		MSE	0.0132	0.0189	0.0165	0.0158
	(r_2, r_2)	Bias	0.0174	0.0340	0.0318	− 0.0290
		MSE	0.0107	0.0179	0.0166	0.0148
	(r_2, r_3)	Bias	0.0133	0.0316	− 0.0255	− 0.0211
		MSE	0.0088	0.0153	0.0120	0.0111
	(r_3, r_3)	Bias	0.0351	0.0499	0.0477	0.0435
		MSE	0.0213	0.0240	0.0243	0.0221
(2, 2.5, 1.5)	(r_1, r_1)	Bias	− 0.0292	0.0369	− 0.0348	− 0.0320
		MSE	0.0204	0.0262	0.0258	0.0232
	(r_1, r_2)	Bias	− 0.0223	0.0398	0.0375	− 0.0340
		MSE	0.0112	0.0183	0.0186	0.0160
	(r_1, r_3)	Bias	− 0.0173	− 0.0254	0.0241	− 0.0220
		MSE	0.0070	0.0109	0.0101	0.0097
	(r_2, r_2)	Bias	− 0.0162	0.0242	0.0237	− 0.0226
		MSE	0.0083	0.0124	0.0111	0.0113
	(r_2, r_3)	Bias	0.0145	0.0257	− 0.0211	− 0.0205
		MSE	0.0123	0.0203	0.0180	0.0176
	(r_3, r_3)	Bias	− 0.0333	0.0503	− 0.0476	− 0.0455
		MSE	0.0172	0.0254	0.0246	0.0225

MLE and Bayes estimates of R over 1,000 replications. All the results are reported in Table 2. From Table 2, it is clear that both of estimators work quite well. We observe that the MLE performs better than the Bayes estimators based on different priors in terms of biases and MSEs. We also observe that there is no significant difference among Bayes estimators based on different values of the prior parameters. Comparing different censoring schemes, we observe that the scheme (r_2, r_3) provides the smallest biases and MSEs for different parameter values.

We also computed the 95% confidence intervals (CIs) for R based on the asymptotic distributions of the MLE and the HPD credible intervals. We have extracted the confidence intervals based on 250 re-sampling for both the bootstrap methods, in our simulation experiments. We presented the average confidence credible lengths, and the corresponding coverage percentages, in Table 3. Comparing the average confidence lengths with the average credible lengths, it is evident that the average credible lengths are less than the average confidence lengths. We also observe that the coverage percentage of the Bayes estimator is more than the coverage percentage of the MLE.

Now we assume that the common scale parameter is known. In this case, the estimates of R are obtained via the MLE, UMVUE, and Bayes methods. Since for the MLE, the exact distribution is also known, therefore, it can be used to construct confidence intervals. The results are presented in Table 4. The results in Table 4 show that the approximate Bayes

Table 3. Average confidence/credible length and coverage percentage (in parentheses) for estimators of R.

(α, β, σ)	C.S.	MLE	Bayes prior 1	Bayes prior 2	Bayes prior 3
(1, 1.5, 0.5)	(r_1, r_1)	0.356(0.873)	0.331(0.930)	0.336(0.932)	0.337(0.937)
	(r_1, r_2)	0.342(0.881)	0.329(0.941)	0.324(0.939)	0.330(0.940)
	(r_1, r_3)	0.345(0.887)	0.323(0.943)	0.330(0.946)	0.327(0.942)
	(r_2, r_2)	0.341(0.892)	0.319(0.950)	0.317(0.939)	0.321(0.947)
	(r_2, r_3)	0.341(0.882)	0.317(0.951)	0.318(0.949)	0.327(0.946)
	(r_3, r_3)	0.351(0.902)	0.330(0.953)	0.329(0.949)	0.321(0.948)
(1, 0.5, 1)	(r_1, r_1)	0.353(0.883)	0.335(0.932)	0.334(0.942)	0.337(0.947)
	(r_1, r_2)	0.345(0.888)	0.319(0.941)	0.323(0.949)	0.320(0.949)
	(r_1, r_3)	0.345(0.897)	0.326(0.945)	0.330(0.956)	0.322(0.952)
	(r_2, r_2)	0.351(0.894)	0.319(0.950)	0.318(0.939)	0.311(0.937)
	(r_2, r_3)	0.341(0.872)	0.317(0.941)	0.316(0.949)	0.328(0.956)
	(r_3, r_3)	0.351(0.910)	0.320(0.953)	0.319(0.939)	0.311(0.948)
(2, 2.5, 1.5)	(r_1, r_1)	0.343(0.894)	0.312(0.956)	0.316(0.948)	0.311(0.953)
	(r_1, r_2)	0.342(0.904)	0.314(0.961)	0.313(0.959)	0.311(0.961)
	(r_1, r_3)	0.332(0.893)	0.309(0.953)	0.312(0.963)	0.310(0.955)
	(r_2, r_2)	0.336(0.904)	0.304(0.956)	0.310(0.949)	0.301(0.961)
	(r_2, r_3)	0.340(0.898)	0.304(0.950)	0.318(0.949)	0.302(0.958)
	(r_3, r_3)	0.350(0.905)	0.334(0.953)	0.335(0.946)	0.321(0.962)

Table 4. Biases and MSE of the MLE, UMVUE, and Bayes estimators of R and average confidence length and coverage percentage when σ is known and $\sigma = 1$.

	C.S.		MLE	UMVUE	Bayes	Exact CI and Cov. prob.
$(\alpha = 1, \beta = 1.5)$	(r_1, r_1)	Bias	−0.0031	0.0016	−0.0076	length 0.3936
		MSE	0.0109	0.0119	0.0100	CP 0.9430
	(r_1, r_2)	Bias	−0.0022	0.0026	−0.0067	length 0.3936
		MSE	0.0107	0.0117	0.0098	CP 0.9530
	(r_1, r_3)	Bias	−0.0043	0.0004	−0.0087	length 0.3938
		MSE	0.0110	0.0120	0.0101	CP 0.9480
	(r_2, r_2)	Bias	−0.0013	0.0034	−0.0060	length 0.3934
		MSE	0.0107	0.0117	0.0098	CP 0.9510
	(r_2, r_3)	Bias	−0.0031	0.0016	−0.0076	length 0.3936
		MSE	0.0109	0.0119	0.0100	CP 0.9430
	(r_3, r_3)	Bias	0.0001	0.0048	−0.0046	length 0.3923
		MSE	0.0111	0.0122	0.0102	CP 0.9420
$(\alpha = 1, \beta = 0.5)$	(r_1, r_1)	Bias	0.0111	0.0122	0.0153	length 0.3726
		MSE	0.0095	0.0101	0.0090	CP 0.9560
	(r_1, r_2)	Bias	0.0075	0.0003	0.0146	length 0.3725
		MSE	0.0094	0.0100	0.0088	CP 0.9530
	(r_1, r_3)	Bias	0.0091	0.0020	0.0161	length 0.3731
		MSE	0.0094	0.0100	0.0089	CP 0.9570
	(r_2, r_2)	Bias	0.0075	0.0003	0.0145	length 0.3721
		MSE	0.0095	0.0102	0.0090	CP 0.9510
	(r_2, r_3)	Bias	0.0083	0.0012	0.0153	length 0.3726
		MSE	0.0095	0.0101	0.0090	CP 0.9560
	(r_3, r_3)	Bias	0.0072	0.0001	0.0141	length 0.3707
		MSE	0.0104	0.0111	0.0098	CP 0.9420
$(\alpha = 2, \beta = 2.5)$	(r_1, r_1)	Bias	−0.0029	−0.0003	−0.0054	length 0.4010
		MSE	0.0124	0.0136	0.0113	CP 0.9400
	(r_1, r_2)	Bias	−0.0052	−0.0028	−0.0075	length 0.4004
		MSE	0.0131	0.0143	0.0119	CP 0.9330
	(r_1, r_3)	Bias	−0.0005	0.0022	−0.0031	length 0.4022
		MSE	0.0113	0.0125	0.0103	CP 0.9490
	(r_2, r_2)	Bias	−0.0024	0.0003	−0.0049	length 0.4020
		MSE	0.0116	0.0128	0.0106	CP 0.9510
	(r_2, r_3)	Bias	−0.0029	−0.0003	−0.0054	length 0.4010
		MSE	0.0124	0.0136	0.0113	CP 0.9400
	(r_3, r_3)	Bias	−0.0045	−0.0020	−0.0069	length 0.4023
		MSE	0.0117	0.0129	0.0107	CP 0.9610

Table 5. Sample median, scale parameter, shape parameter, K-S, and *p* value of the fitted GL distributions to datasets.

Dataset	Sample median	Scale parameter	Shape parameter	K-S	*p* Value
1	2.996	0.3378	0.9036	0.1151	0.3471
2	2.478	0.2856	1.0652	0.0791	0.7508

Table 6. Observed frequencies and expected frequencies for modified dataset I when fitting the GL distribution.

Intervals	Observed frequencies	Expected frequencies	Chi-square
< 2.5	12	10.7938	0.8406
2.5–3	20	18.7095	
3–3.5	17	19.8368	
3.5–4	9	9.5272	
> 4	5	4.1328	

estimator provides the smallest MSEs. Comparing different censoring schemes, we observe that the scheme (r_2, r_2) provides the smallest biases and MSEs.

5. Data analysis

Here, analysis of the strength data, which was originally reported by Badar and Priest (1982) is presented. The GL distribution models for the two datasets are fitted separately. The estimated scale and shape parameters are proposed assuming the location parameter to be known as the sample median for both the datasets. We also obtained Kolmogorov–Smirnov (K-S) distance between the empirical distribution functions, and the fitted distributions, and corresponding *p* values. All the results have reported in Table 5. For comparison purposes, we also compute the observed and the expected frequencies, the corresponding chi-square values based on the fitted models in Tables 6 and 7.

It is clear that the GL distribution models fit quite well to both the datasets. For both datasets, the fitted empirical cdf plots of the GL distribution model are shown in Figs. 3 and 4. Figures 3 and 4 indicate a satisfactory fit for the GL distribution model. Considering the results of Table 5, because the two scale parameters are not very different, assuming the two parameters are equal, $\hat{R} = 0.5630$ and the approximate confidence interval of R, (0.4788,0.6472) is derived for completed data.

For illustrative purposes, we have generated two different progressively censored samples using two different sampling schemes from Table 8. The generated data and corresponding censored schemes have been presented in Table 9. From (3) and (19), the ML and Bayes estimates of R become 0.5477 and 0.5482, respectively. To compute the Bayes estimate, we adopted the suggestion of Congdon (2001) and Kundu and Gupta (2005), that is, $a_1 = a_2 = a_3 = b_1 =$

Table 7. Observed frequencies and expected frequencies for modified dataset II when fitting the GL distribution.

Intervals	Observed frequencies	Expected frequencies	Chi-square
<1.76	5	5.4891	0.6465
1.76–2.22	15	15.8443	
2.22–2.68	27	26.8324	
2.68–3.14	18	15.9098	
>3.14	4	5.2843	

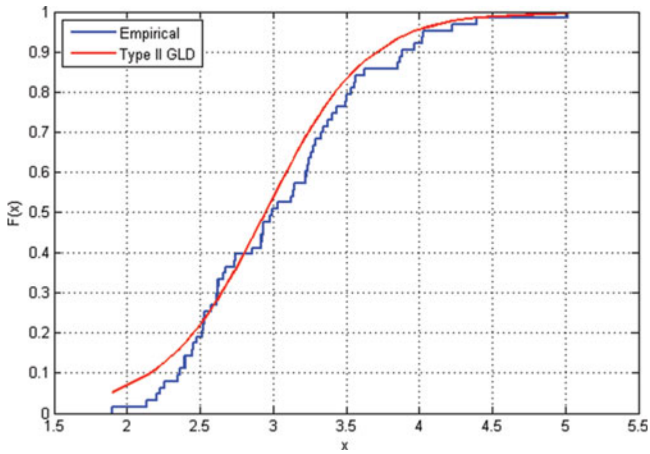

Figure 3. The fitted empirical cdf plots of the GL distribution (Dataset I).

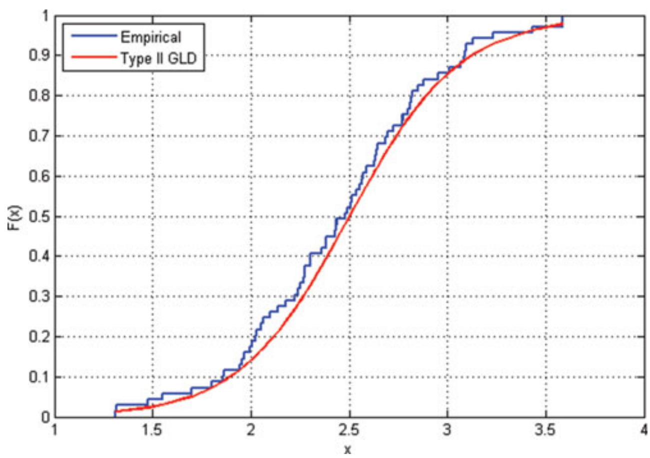

Figure 4. The fitted empirical cdf plots of the GL distribution (Dataset II).

Table 8. The real datasets.

Dataset I (x)							Dataset II (y)						
1.901	2.132	2.203	2.228	2.257	2.350	2.361	1.312	1.314	1.479	1.552	1.700	1.803	1.861
2.396	2.397	2.445	2.454	2.474	2.518	2.522	1.865	1.944	1.958	1.966	1.997	2.006	2.021
2.525	2.532	2.575	2.614	2.616	2.618	2.624	2.027	2.055	2.063	2.098	2.140	2.179	2.224
2.659	2.675	2.738	2.740	2.856	2.917	2.928	2.240	2.253	2.270	2.272	2.274	2.301	2.301
2.937	2.937	2.977	2.996	3.030	3.125	3.139	2.359	2.382	2.382	2.426	2.434	2.435	2.478
3.145	3.220	3.223	3.235	3.243	3.264	3.272	2.490	2.511	2.514	2.535	2.554	2.566	2.570
3.294	3.332	3.346	3.377	3.408	3.435	3.493	2.586	2.629	2.633	2.642	2.648	2.684	2.697
3.501	3.537	3.554	3.562	3.628	3.852	3.871	2.726	2.770	2.773	2.800	2.809	2.818	2.821
3.886	3.971	4.024	4.027	4.225	4.395	5.020	2.848	2.880	2.954	3.012	3.067	3.084	3.090
							3.096	3.128	3.233	3.433	3.585	3.585	

Table 9. The generated data and corresponding censored schemes.

i	1	2	3	4	5	6	7	8	9	10
x_i	1.901	2.361	2.518	2.616	2.740	2.977	3.320	3.394	3.493	3.852
r_i	5	5	5	5	5	5	5	5	5	8
y_j	1.312	1.861	2.006	2.140	2.272	2.382	2.511	2.586	2.697	2.818
r'_i	5	5	5	5	5	5	5	5	5	14

$b_2 = b_3 = 0.0001$. The 95% confidence intervals corresponding ML and Bayes estimators of R become (0.3447, 0.7508) and (0.3349,0.7489), respectively. The 95% percentile bootstrap method (see Efron, 1982) and bootstrap-t method (see Hall, 1988) confidence intervals are obtained as (0.3319,0.7463) and (0.3407, 0.7513), respectively. We observe that the results are not significantly different from the corresponding results obtained from completed data.

6. Concluding remarks

In the previous sections, estimation of $R = P(X < Y)$ when X and Y come from two independent GL distributions of Type-II with different parameters, based on progressively Type-II censored samples was discussed. Now, we can extend our methods for a much larger class of distribution functions, namely, proportional hazard rate models (PHRM). The following definitions are required for continuing the article:

Definition 6.1. Let X be an absolutely continuous random variable with distribution function $F(.)$ and hazard rate function $h(.)$. The family of random variables with hazard rate function of the form $\{\alpha h(.) : \alpha > 0\}$ is called the proportional hazard rate family and the distribution function $F(.)$ is called the baseline distribution function of that family.

Therefore, a one-dimensional PHRM is a parametric family of the distribution function:

$$F_{\mathrm{PHRM}}(x; \alpha, \sigma) = 1 - [\bar{F}(x; \sigma)]^{\alpha}, \qquad x \in S_X, \tag{36}$$

where $\bar{F}(x; \sigma) = 1 - F(x; \sigma)$ is the survival function of the baseline distribution $F(x; \sigma)$ with support S_X and $\alpha > 0$ and $\sigma > 0$ are shape and scale parameters, respectively. The pdf corresponding to (36) is

$$f_{\mathrm{PHRM}}(x; \alpha, \sigma) = \alpha[\bar{F}(x; \sigma)]^{\alpha-1} f(x; \sigma), \qquad x \in S_X.$$

This model, commonly known as Cox's proportional hazard, was introduced by Lehmann (1953). Applications of hazard rate functions are quite well-known in the statistical literature. PHRM is used to model failure time data in survival analysis. This family includes several well-known lifetime distributions such as Weibull (one parameter), generalized Pareto, generalized logistic Type-II, Lomax, Burr Type-XII and so on.

6.1. Maximum likelihood estimator

In this section, we obtain the MLE of R, when the common scale parameter σ, is the same. Let $X \sim \mathrm{PHRM}(\alpha, \sigma)$ and $Y \sim \mathrm{PHRM}(\beta, \sigma)$, where X and Y are independent random variables. It is easy to show that

$$R = P(X < Y) = \int_{-\infty}^{+\infty} f_{\mathrm{PHRM}}(y; \beta, \sigma)[1 - (\bar{F}(y; \alpha, \sigma))^{\alpha}]dy = \frac{\beta}{\alpha + \beta}.$$

So, in order to obtain the MLE of R, we need to compute the MLE of α and β. Suppose $\mathbf{X} = (X_{1:m_1:n_1}, X_{2:m_1:n_1}, \ldots, X_{m_1:m_1:n_1})$ is a progressively Type-II censored sample from $\mathrm{PHRM}(\alpha, \sigma)$ with censored scheme $\mathbf{r} = (r_1, r_2, \ldots, r_{m_1})$ and $\mathbf{Y} = (Y_{1:m_2:n_2}, Y_{2:m_2:n_2}, \ldots, Y_{m_2:m_2:n_2})$ is also a progressively Type-II censored sample from $\mathrm{PHRM}(\beta, \sigma)$ with censored scheme $\mathbf{r}' = (r'_1, r'_2, \ldots, r'_{m_2})$. Therefore, the log-likelihood function of the observed sample is

$$l(\alpha,\beta,\sigma) = c - m_1 \ln\sigma + m_1 \ln\alpha - \sum_{i=1}^{m_1} \ln f\left(\frac{x_{i:m_1:n_1}}{\sigma}\right) - \sum_{i=1}^{m_1} (\alpha r_i + \alpha + 1) \ln \bar{F}\left(\frac{x_{i:m_1:n_1}}{\sigma}\right)$$
$$- m_2 \ln\sigma + m_2 \ln\beta - \sum_{i=1}^{m_2} \ln f\left(\frac{y_{i:m_2:n_2}}{\sigma}\right) - \sum_{i=1}^{m_2} (\beta r_i' + \beta + 1) \ln \bar{F}\left(\frac{y_{i:m_2:n_2}}{\sigma}\right).$$

In this case, we find

$$\hat{\alpha}_{(\hat{\sigma})} = -\frac{m_1}{\sum_{i=1}^{m_1} (r_i + 1) \ln \bar{F}\left(\frac{X_{i:m_1:n_1}}{\hat{\sigma}}\right)}, \tag{37}$$

and

$$\hat{\beta}_{(\hat{\sigma})} = -\frac{m_2}{\sum_{i=1}^{m_1} (r_i' + 1) \ln \bar{F}\left(\frac{Y_{i:m_2:n_2}}{\hat{\sigma}}\right)}. \tag{38}$$

where $\hat{\sigma}$ can be achieved by using an iterative scheme explained in Section 3. By using the invariance property of MLE, we conclude that

$$\hat{R} = \frac{\hat{\beta}_{(\hat{\sigma})}}{\hat{\alpha}_{(\hat{\sigma})} + \hat{\beta}_{(\hat{\sigma})}}.$$

Now if the scale parameter σ be known, in this case the MLE of R becomes

$$\hat{R} = \frac{\hat{\beta}}{\hat{\alpha} + \hat{\beta}} = \frac{1}{1 + \frac{m_1 T_2}{m_2 T_1}},$$

where $T_1 = -\sum_{i=1}^{m_1} (r_i + 1) \ln \bar{F}(\frac{X_{i:m_1:n_1}}{\sigma})$ and $T_2 = -\sum_{i=1}^{m_1} (r_i' + 1) \ln \bar{F}(\frac{Y_{i:m_2:n_2}}{\sigma})$.

Similarly in Section 4, it is easy to show that $2\alpha T_1$ and $2\beta T_2$ have chi-square distribution with $2m_1$ and $2m_2$ degrees of freedom, respectively. Therefore, the random variable $F = \frac{\beta}{\alpha}(\frac{1}{\hat{R}} - 1)$ has an $F_{2m_2,2m_1}$ distribution with $2m_2$ and $2m_1$ degrees of freedom. By using this fact, we can extract the exact distribution of $\hat{R}$ and the $100(1-\gamma)\%$ confidence interval of R as (25) and (26), respectively.

6.2. Bayes estimation

Here, we try to find the Bayes estimator R under the assumption that the shape parameters α and β and the scale parameter σ are random variables. It is assumed that α, β, and σ have independent gamma priors with the parameters $\alpha \sim \text{Gamma}(a_1, b_1)$ and $\sigma \sim \text{Gamma}(a_2, b_2)$ with the pdfs (14), (15), and (16). The posterior pdfs of α and β are as follows:

$$\alpha\,|\sigma, \text{data} \sim \text{Gamma}\left(a_1 + m_1, b_1 - \sum_{i=1}^{m_1} (r_i + 1) \ln \bar{F}\left(\frac{X_{i:m_1:n_1}}{\sigma}\right)\right),$$

and

$$\beta\,|\sigma, \text{data} \sim \text{Gamma}\left(a_2 + m_2, b_2 - \sum_{i=1}^{m_2} (r_i' + 1) \ln \bar{F}\left(\frac{Y_{i:m_2:n_2}}{\sigma}\right)\right).$$

The posterior pdf of σ is not known, therefore we use the Gibbs sampling technique to get the Bayes estimate of α and σ corresponding credible interval explained in Section 3.

6.3. UMVUE

When the common scale parameter σ is known, (T_1, T_2) is a jointly complete sufficient statistic for (α, β). Let us define

$$\phi(X_{1:m_1:n_1}, Y_{1:m_2:n_2}) = \begin{cases} 1 & \text{if} \quad -n_2 \ln \bar{F}(Y_{1:m_2:n_2}) < -n_1 \ln \bar{F}(X_{1:m_1:n_1}) \\ 0 & \text{if} \quad -n_2 \ln \bar{F}(Y_{1:m_2:n_2}) > -n_1 \ln \bar{F}(X_{1:m_1:n_1}). \end{cases} \tag{39}$$

It is easy to show that, $\phi(X_{1:m_1:n_1}, Y_{1:m_2:n_2})$ is an unbiased estimator of R. Applying same argument used in Section 4, the UMVUE of R is extracted as

$$\tilde{R} = \begin{cases} 1 - \sum_{i=0}^{m_2-1} (-1)^i \frac{(m_2-1)!(m_1-1)!}{(m_2-i-1)!(m_1+i-1)!} (\frac{T_1}{T_2})^i & \text{if} \quad T_1 \leq T_2 \\ \sum_{i=0}^{m_1-1} (-1)^i \frac{(m_1-1)!(m_2-1)!}{(m_1+i-1)!(m_2-i-1)!} (\frac{T_2}{T_1})^i & \text{if} \quad T_2 \leq T_1, \end{cases} \tag{40}$$

where $T_1 = -\sum_{i=1}^{m_1} (r_i + 1) \ln \bar{F}(\frac{X_{i:m_1:n_1}}{\sigma})$ and $T_2 = -\sum_{i=1}^{m_1} (r'_i + 1) \ln \bar{F}(\frac{Y_{i:m_2:n_2}}{\sigma})$.

Stress–strength reliability can also be defined in a more general multicomponent form (see Bhattacharyya and Johnson, 1974; Chandra and Owen, 1977; Eryilmaz, 2008). For example, a series system consists of n components whose random strengths are denoted by $X_1, X_2, \ldots, X_n$. In this case, the reliability of the system corresponds to $R = P(Y < \min(X_1, X_2, \ldots, X_n))$.

When α is an integer, (36) has an interpretation familiar in reliability theory; it is the survival function of a series system of α independent components each with survival function $\bar{F}(.)$. Now, let $X_1, X_2, \ldots, X_\alpha$ be the lifetime of independent components (random strengths) of the series system with α components and survival function $\bar{F}(.)$ and $Y \sim \text{PHRM}(\beta)$ is stress, then the reliability of the system is resulted as follows:

$$R = P(Y < \min(X_1, X_2, \ldots, X_n)) = \frac{\beta}{\alpha + \beta}.$$

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