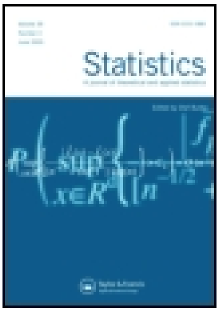

Statistics: A Journal of Theoretical and Applied Statistics
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/gsta20

Inference of R=P[X<Y] for generalized logistic distribution

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Version of record first published: 22 Jan 2013.

To cite this article: Salman Babayi, Esmaile Khorram & Farzad Tondro (2013): Inference of R=P[X<Y] for generalized logistic distribution, Statistics: A Journal of Theoretical and Applied Statistics, DOI:10.1080/02331888.2012.760095

To link to this article: http://dx.doi.org/10.1080/02331888.2012.760095

Statistics, 2013
http://dx.doi.org/10.1080/02331888.2012.760095

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Inference of $R = P[X < Y]$ for generalized logistic distribution

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(*Received 7 December 2011; final version received 15 December 2012*)

This paper deals with the estimation of $R = P[X < Y]$ when X and Y come from two independent generalized logistic distributions with different parameters. The maximum-likelihood estimator (MLE) and its asymptotic distribution are proposed. The asymptotic distribution is used to construct an asymptotic confidence interval of R. Assuming that the common scale parameter is known, the MLE, uniformly minimum variance unbiased estimator, Bayes estimation and confidence interval of R are obtained. The MLE of R, asymptotic distribution of R in the general case, is also discussed. Monte Carlo simulations are performed to compare the different proposed methods. Analysis of a real data set has also been presented for illustrative purposes.

Keywords: generalized logistic distribution; maximum-likelihood estimator; Bayes estimator; Monte Carlo simulations

1. Introduction

The problem of making inference about $R = P[X < Y]$ has received considerable attention in the literature. This problem arises naturally in the context of mechanical reliability of a system with stress X and strength Y. The system fails if at any time the applied stress is greater than its strength.

Various versions of this problem have been discussed in the literature. The maximum-likelihood estimator (MLE) of $P[Y < X]$, when X and Y are normally distributed, has been considered by Downtown [1]. Tong [2] derived the minimum variance unbiased estimator of R when X and Y are independent exponential random variables. The gamma case has been studied by Constantine and Karson [3]. Recently, inferences on reliability in the two-parameter exponential stress–strength model [4] and estimation of $P[Y < X]$ from logistic [5], Laplace [6], bivariate exponential [7] and Weibull [8] distributions have been discussed also. A comprehensive treatment of the different stress–strength models can be found in the recent monograph [9].

The main aim of this paper is to discuss the inference of $P[Y < X]$, when X and Y have two independent generalized logistic (GL) distribution with different parameters. In Section 2, the logistic distribution is introduced. In Section 3, we drive the MLE of R. The asymptotic distribution of the MLE of R is given and asymptotic confidence interval is proposed. In Section 4, we consider different estimations of R when the common scale parameter is known. In this section, the MLE,

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the uniformly minimum variance unbiased estimator (UMVUE) and the Bayes estimation of R are discussed. In Section 5, the estimation of R in the general case is studied. The MLE of R and asymptotic distribution are presented in this section. The different proposed methods have been compared using Monte Carlo simulations and their results have been reported in Section 6. Finally, a data analysis has been done in Section 7.

2. GL distribution

The random variable X has the GL distribution if it has the following cumulative distribution function:

$$F(x;\mu,\sigma,\alpha)=\frac{1}{(1+\mathrm{e}^{-(x-\mu)/\sigma})^{\alpha}},\quad -\infty<x<+\infty,$$

where $\mu\in\mathbb{R}$ and $\sigma,\alpha\in(0,+\infty)$. Here, μ, σ and α are the location, scale and shape parameters, respectively. In the particular case of $\alpha=1$, F corresponds to the usual logistic distribution. Zelterman [10] and George, and Ojo [11] studied several useful applications of the GL distribution. Other generalizations of the logistic distribution have been discussed by Johnson and Kotz [12]. Zelterman [13] showed that the maximum-likelihood estimates do not exist for (μ,σ,α). Therefore, for convenience, we suppose that $\mu=0$.

3. MLE of *R* with a common scale parameter

In this section, we investigate the properties of R, when the common scale parameter σ is the same.

Let $X\sim\mathrm{GL}(\alpha,\sigma)$ and $Y\sim\mathrm{GL}(\beta,\sigma)$, where X and Y are independent random variables. Therefore,

$$R=P(X<Y)=\frac{\beta}{\alpha+\beta}.$$

So, in order to obtain the MLE of R, we need to compute the MLE of α and β. Suppose $X_1,X_2,\ldots,X_n$ is a random sample from $\mathrm{GL}(\alpha,\sigma)$ and $Y_1,Y_2,\ldots,Y_m$ is also a random sample from $\mathrm{GL}(\beta,\sigma)$. Therefore, the log-likelihood function of the observed sample is

$$\begin{aligned}L(\mu,\sigma,\alpha)&=n\ln\alpha-\frac{1}{\sigma}\sum_{i=1}^{n}x_i-n\ln\sigma-(\alpha+1)\sum_{i=1}^{n}\ln(1+\mathrm{e}^{-x_i/\sigma})\\&\quad+m\ln\beta-\frac{1}{\sigma}\sum_{i=1}^{m}y_i-m\ln\sigma-(\beta+1)\sum_{i=1}^{m}\ln(1+\mathrm{e}^{-y_i/\sigma}).\end{aligned}$$

The MLEs of α, β and σ, say $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\sigma}$, respectively, can be achieved as

$$\hat{\alpha}=\frac{n}{\sum_{i=1}^{n}\ln(1+\mathrm{e}^{-X_i/\hat{\sigma}})},\tag{1}$$

$$\hat{\beta}=\frac{m}{\sum_{i=1}^{m}\ln(1+\mathrm{e}^{-Y_i/\hat{\sigma}})},\tag{2}$$

and $\hat{\sigma}$ can be given as the solution of the nonlinear equation

$$\frac{1}{\sigma^2}\left(\sum_{i=1}^{n} x_i + \sum_{i=1}^{m} y_i\right) - \frac{m+n}{\sigma} - \frac{1}{\sigma^2}\left(\frac{n}{\sum_{i=1}^{n}\ln(1+\mathrm{e}^{-x_i/\sigma})} + 1\right)\sum_{i=1}^{n}\frac{x_i\,\mathrm{e}^{-x_i/\sigma}}{1+\mathrm{e}^{-x_i/\sigma}}$$
$$-\frac{1}{\sigma^2}\left(\frac{m}{\sum_{i=1}^{m}\ln(1+\mathrm{e}^{-y_i/\sigma})} + 1\right)\sum_{i=1}^{m}\frac{y_i\,\mathrm{e}^{-y_i/\sigma}}{1+\mathrm{e}^{-y_i/\sigma}} = 0. \tag{3}$$

By the invariance property of MLEs, the MLE of R becomes

$$\hat{R} = \frac{\hat{\beta}}{\hat{\alpha}+\hat{\beta}}. \tag{4}$$

3.1. *Asymptotic distribution*

In this section, first the asymptotic distribution of $\hat{\theta} = (\hat{\alpha}, \hat{\beta}, \hat{\sigma})$ is considered, then the asymptotic distribution of $\hat{R}$ is extracted. Based on the asymptotic distribution of $\hat{R}$, the asymptotic confidence interval of R is managed. We denote the Fisher information matrix of $\theta = (\alpha, \beta, \sigma)$ as $I(\theta) = [I_{ij}(\theta)]$, $i,j = 1,2,3$. Therefore,

$$I(\theta) = -\begin{bmatrix} E\left(\frac{\partial^2 L}{\partial\alpha^2}\right) & E\left(\frac{\partial^2 L}{\partial\alpha\partial\beta}\right) & E\left(\frac{\partial^2 L}{\partial\alpha\partial\sigma}\right) \\ E\left(\frac{\partial^2 L}{\partial\beta\partial\alpha}\right) & E\left(\frac{\partial^2 L}{\partial\beta^2}\right) & E\left(\frac{\partial^2 L}{\partial\beta\partial\sigma}\right) \\ E\left(\frac{\partial^2 L}{\partial\sigma\partial\alpha}\right) & E\left(\frac{\partial^2 L}{\partial\sigma\partial\beta}\right) & E\left(\frac{\sigma^2 L}{\partial\sigma^2}\right) \end{bmatrix} = \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix}.$$

THEOREM 1 [14,15] *As $n \to \infty$, $m \to \infty$ and $n/m \to p$, then*

$$[\sqrt{n}(\hat{\alpha}-\alpha), \sqrt{m}(\hat{\beta}-\beta), \sqrt{n}(\hat{\sigma}-\sigma)] \to N_3(0, U^{-1}(\alpha,\beta,\sigma)),$$

where

$$U(\alpha,\beta,\sigma) = \begin{bmatrix} u_{11} & 0 & u_{13} \\ 0 & u_{22} & u_{23} \\ u_{31} & u_{32} & {}_{33} \end{bmatrix},$$

and considering the results of Zelterman [13], u_{ij} *is given by*

$$u_{11} = \frac{1}{n}I_{11} = \frac{1}{\alpha^2},$$
$$u_{13} = u_{31} = \frac{1}{n}I_{13} = \frac{1}{\sigma(\alpha+1)}(\psi(\alpha) - \psi(2)),$$
$$u_{22} = \frac{1}{m}I_{22} = \frac{1}{\beta^2},$$

$$u_{23} = u_{32} = \frac{\sqrt{p}}{n} I_{23} = \frac{1}{\sqrt{p}} \frac{1}{\sigma(\beta+1)} (\psi(\beta) - \psi(2)),$$

$$u_{33} = \frac{1}{n} I_{33} = \frac{1}{\sigma^2} + \frac{\alpha}{\sigma^2(\alpha+2)} [\psi'(\alpha+1) + \psi'(2) + (\psi(\alpha+1) - \psi(2))^2]$$

$$+ \frac{1}{p\sigma^2} + \frac{\beta}{p\sigma^2(\beta+2)} [\psi'(\beta+1) + \psi'(2) + (\psi(\beta+1) - \psi(2))^2].$$

Proof The proof follows from the asymptotic normality of the MLE. ■

THEOREM 2 *As $n \to \infty$ and $m \to \infty$ so that $n/m \to p$, then*

$$\sqrt{n}(\hat{R} - R) \to N(0, B),$$

where

$$B = \frac{1}{k(\alpha+\beta)^4} [\beta^2(u_{22}u_{33} - u_{23}^2) - 2\alpha\beta\sqrt{p} u_{23}u_{31} + \alpha^2 p(u_{11}u_{33} - u_{13}^2)]$$

and

$$k = u_{11}u_{22}u_{33} - u_{11}u_{23}u_{32} - u_{13}u_{22}u_{31}.$$

Proof By using Theorem 1 and Slutsky's theorem, the consistency and asymptotic normality of the MLE, the proof will be completed. ■

Using Theorem 2, the asymptotic confidence interval of R is given by

$$\left(\hat{R} - z_{1-\gamma/2} \frac{\sqrt{\hat{B}}}{\sqrt{n}}, \hat{R} + z_{1-\gamma/2} \frac{\sqrt{\hat{B}}}{\sqrt{n}} \right).$$

To estimate variance B, the empirical Fisher information matrix and MLE of α, β and σ may be used.

4. Estimation of *R* if *σ* is known

In this section, the estimation of R when σ is known is considered. Without loss of generality, we assume that $\sigma = 1$.

4.1. *MLE of R*

Let $X_1, X_2, \ldots, X_n$ be a random sample from $\mathrm{GL}(\alpha, 1)$ and $Y_1, Y_2, \ldots, Y_m$ be a random from $\mathrm{GL}(\beta, 1)$. Based on Section 3, it is clear that the MLE of R will be

$$\hat{R} = \frac{m \sum_{i=1}^{n} \ln(1 + \mathrm{e}^{-X_i})}{n \sum_{i=1}^{m} \ln(1 + \mathrm{e}^{-Y_i}) + m \sum_{i=1}^{n} \ln(1 + \mathrm{e}^{-X_i})}.$$

The $100(1-\gamma)\%$ confidence interval of R can be obtained as

$$\left[\frac{1}{1 + F_{2n,2m;1-\gamma/2}(1/\hat{R} - 1)}, \frac{1}{1 + F_{2n,2m;\gamma/2}(1/\hat{R} - 1)} \right],$$

where $F_{2n,2m;\gamma/2}$ and $F_{2n,2m;1-\gamma/2}$ are the lower and upper $\gamma/2$th percentile points of an F distribution with $2n$ and $2m$ degrees of freedom.

4.2. *UMVUE of R*

Since $\ln(1+e^{-X_i})$ and $\ln(1+e^{-Y_i})$ are the exponential random variables, therefore applying the results of Tong [2,16], the $\tilde{R}$ is given by

$$\tilde{R} = \begin{cases} 1 - \sum_{i=0}^{m-1} (-1)^i \dfrac{(n-1)!(m-1)!}{(m-i-1)!(n+i-1)!} \left(\dfrac{T_1}{T_2}\right)^i & \text{if } T_1 \leq T_2, \\ \sum_{i=0}^{n-1} (-1)^i \dfrac{(n-1)!(m-1)!}{(m+i-1)!(n-i-1)!} \left(\dfrac{T_2}{T_1}\right)^i & \text{if } T_2 \leq T_1, \end{cases}$$

where $T_1 = \sum_{i=1}^{n} \ln(1+e^{-X_i})$ and $T_2 = \sum_{i=1}^{m} \ln(1+e^{-Y_i})$.

4.3. *Bayes estimator and credible interval of R*

In this subsection, we attempt to find the Bayes estimator of R under the assumption that the shape parameters α and β are random variables. It is assumed that α and β have independent gamma priors with the parameters $\alpha \sim \text{Gamma}(a_1, b_1)$ and $\beta \sim \text{Gamma}(a_2, b_2)$.

The posterior probability density functions (PDFs) of α and β are as follows:

$$\alpha|\text{data} \sim \text{Gamma}(a_1 + n, b_1 + T_1), \tag{5}$$

$$\beta|\text{data} \sim \text{Gamma}(a_2 + m, b_2 + T_2), \tag{6}$$

where $T_1 = \sum_{i=1}^{n} \ln(1+e^{-X_i})$ and $T_2 = \sum_{i=1}^{m} \ln(1+e^{-Y_i})$. Since the prior distributions α and β are independent, with Equations (5) and (6), the posterior PDF of R becomes

$$f_R(r) = C \frac{r^{a_2+m-1}(1-r)^{a_1+n-1}}{((b_1+T_1)(1-r)+(b_2+T_2)r)^{m+n+a_1+a_2}}, \quad 0 < r < 1, \tag{7}$$

where

$$C = \frac{\Gamma(m+n+a_1+a_2)}{\Gamma(n+a_1)\Gamma(m+a_2)} (b_1+T_1)^{a_1+n}(b_2+T_2)^{a_2+m},$$

since the Bayes estimate of R under the squared error loss function cannot be obtained analytically, so, we approximate it via the method of Lindley [17]. Alternatively, via the approximation of Lindley [17] and following the approach of Ahmad *et al.* [18], it can be seen that the approximate Bayes estimate of R, say $\hat{R}_{\text{Bayes}}$, under the squared error loss function is

$$\hat{R}_{\text{Bayes}} = \tilde{R}\left[1 + \frac{\tilde{\alpha}\tilde{R}^2(\tilde{\alpha}(n+a_1-1) - \tilde{\beta}(m+a_2-1))}{\tilde{\beta}^2(n+b_1-1)(m+a_2-1)}\right],$$

where

$$\tilde{R} = \frac{\tilde{\beta}}{\tilde{\alpha}+\tilde{\beta}}, \quad \tilde{\alpha} = \frac{n+a_1-1}{b_1+T_1} \quad \text{and} \quad \tilde{\beta} = \frac{m+a_2-1}{b_2+T_2}.$$

Now, we illustrate briefly how to obtain the approximate highest posterior density (HPD) credible interval of R using the method proposed by Chen and Shao [19].

Step 1: Generate $\{R_1, R_2, \ldots, R_n\}$ from the posterior density function (7) the by accept–reject sampling algorithm.

Step 2: Let $R_{(1)} < R_{(2)} < \cdots < R_{(n)}$ be the ordered R_i, and suppose we would like to construct a $100(1-\gamma)\%$ approximate HPD credible interval of R, then consider the following:

$$(R_{(1)}, R_{((1-\beta)M)}), \ldots, (R_{(\beta M)}, R_{(M)}).$$

Choose that interval which has the shortest length.

5. Estimation of *R* in general case

In this section, we consider the estimation of R when the scale parameters are different.

5.1. *MLE of R*

Let $X \sim \mathrm{GL}(\alpha, \sigma_1)$ and $Y \sim \mathrm{GL}(\beta, \sigma_2)$, where X and Y are independent random variables. Therefore,

$$\begin{aligned} R = P(X < Y) &= \int_{-\infty}^{+\infty} \int_{-\infty}^{y} \frac{\alpha\, \mathrm{e}^{-x/\sigma_1}}{\sigma_1 (1+\mathrm{e}^{-x/\sigma_1})^{\alpha+1}} \cdot \frac{\beta\, \mathrm{e}^{-y/\sigma_2}}{\sigma_2 (1+\mathrm{e}^{-y/\sigma_2})^{\beta+1}}\, \mathrm{d}x\, \mathrm{d}y \\ &= \int_{-\infty}^{+\infty} \frac{1}{(1+\mathrm{e}^{-y/\sigma_1})^{\alpha}} \frac{\beta\, \mathrm{e}^{-y/\sigma_2}}{\sigma_2 (1+\mathrm{e}^{-y/\sigma_2})^{\beta+1}}\, \mathrm{d}y \\ &= \int_{-\infty}^{+\infty} g(y) f_Y(y)\, \mathrm{d}y = \mathrm{E}(g(Y)), \end{aligned}$$

where $g(Y) = 1/(1+\mathrm{e}^{-y/\sigma_1})^{\alpha}$.

Since, the estimation of $R = \mathrm{E}(g(Y))$ cannot be calculated analytically, so we approximated it through a simulation method.

To compute the MLE of R, suppose $X_1, X_2, \ldots, X_n$ is a random sample from $\mathrm{GL}(\alpha, \sigma_1)$ and $Y_1, Y_2, \ldots, Y_m$ is also a random sample from $\mathrm{GL}(\beta, \sigma_2)$. Therefore, the log-likelihood function of the observed sample is

$$\begin{aligned} L(\alpha, \beta, \sigma_1, \sigma_2) = n \ln \alpha - \frac{1}{\sigma_1} \sum_{i=1}^{n} x_i - n \ln \sigma_1 - (\alpha+1) \sum_{i=1}^{n} \ln(1+\mathrm{e}^{-x_i/\sigma_1}) \\ + m \ln \beta - \frac{1}{\sigma_2} \sum_{i=1}^{m} y_i - m \ln \sigma_2 - (\beta+1) \sum_{i=1}^{m} \ln(1+\mathrm{e}^{-y_i/\sigma_2}). \end{aligned}$$

The MLE of α, β, σ_1 and σ_2, say $\hat{\alpha}$, $\hat{\beta}$, $\hat{\sigma}_1$ and $\hat{\sigma}_2$, respectively, will be found as

$$\hat{\alpha} = \frac{n}{\sum_{i=1}^{n} \ln(1+\mathrm{e}^{-x_i/\hat{\sigma}_1})}, \tag{8}$$

$$\hat{\beta} = \frac{m}{\sum_{i=1}^{m} \ln(1+\mathrm{e}^{-y_i/\hat{\sigma}_2})}, \tag{9}$$

and $\hat{\sigma}_1$ and $\hat{\sigma}_2$ will be obtained as the solution of the nonlinear equations

$$\frac{1}{\sigma_1^2} \sum_{i=1}^{n} x_i - \frac{n}{\sigma_1} - \frac{1}{\sigma_1^2} \left(\frac{n}{\sum_{i=1}^{n} \ln(1+\mathrm{e}^{-x_i/\hat{\sigma}_1})} + 1 \right) \sum_{i=1}^{n} \frac{x_i\, \mathrm{e}^{-x_i/\sigma_1}}{(1+\mathrm{e}^{-x_i/\sigma_1})} = 0 \tag{10}$$

and

$$\frac{1}{\sigma_2^2}\sum_{i=1}^{m} y_i - \frac{m}{\sigma_2} - \frac{1}{\sigma_2^2}\left(\frac{m}{\sum_{i=1}^{m}\ln(1+\mathrm{e}^{-y_i/\hat{\sigma}_2})}+1\right)\sum_{i=1}^{m}\frac{y_i\,\mathrm{e}^{-y_i/\sigma_2}}{(1+\mathrm{e}^{-y_i/\sigma_2})} = 0. \tag{11}$$

By the invariance property of the MLEs, the MLE of R becomes

$$\hat{R} = \int_{-\infty}^{+\infty} \frac{1}{(1+\mathrm{e}^{-y/\hat{\sigma}_1})^{\hat{\alpha}}}\frac{\hat{\beta}\,\mathrm{e}^{-y/\hat{\sigma}_2}}{\hat{\sigma}_2(1+\mathrm{e}^{-y/\hat{\sigma}_2})^{\hat{\beta}+1}}\,\mathrm{d}y. \tag{12}$$

5.2. *Asymptotic distribution*

The asymptotic distribution of $\hat{\theta} = (\hat{\alpha}, \hat{\beta}, \hat{\sigma}_1, \hat{\sigma}_2)$ is resulted and similar to Theorems 1 and 2, the asymptotic distribution of $\hat{R}$ can be attained. Let us denote the Fisher information matrix of $\theta = (\alpha, \beta, \sigma_1, \sigma_2)$ as $I(\theta) = [I_{ij}(\theta)],\ i,j = 1,2,3,4$, where $I_{ij} = -\mathrm{E}(\partial^2 L/\partial\theta_i\partial\theta_j)$. Applying the results of Zelterman [13], I_{ij} is resulted. Based on the above Fisher information matrix, it is possible to present confidence intervals of R. But, since it is similar to that mentioned in Section 3, we omit it here.

6. Simulation results

In this section, we mainly present some Monte Carlo simulation results, we performed to observe the behaviour of the different methods for different sample sizes, and for different parameter values. Three cases separately are considered to draw inference on R, namely when (i) σ is unknown, (ii) σ is known and (iii) σ_1 and σ_2 are unknown. In the first two cases, we study the following small sample sizes: $(m, n) = (15, 15), (25, 25), (25, 50), (50, 50)$. We select different values for α, β and σ, also.

In the first case, the common scale parameter σ is considered to be unknown. From the sample, the estimate of σ is computed from Equation (3) via the iterative algorithm. The iterative process stops when the difference between the two consecutive iterates become less than 10^{-6}. Once the σ was estimated, then α and β are estimated using Equations (1) and (2), respectively.

Finally, the MLE of R is resulted using Equation (4). The average biases and MSE of $\hat{R}$ over 1000 replications is reported. Also, the 95% confidence intervals is computed based on the asymptotic distribution of $\hat{R}$ through the estimation of B using two different methods: (i) applying Theorem 2 and replacing the parameters by their estimates (denoting them by $\mathrm{CI_e}$) and (ii) estimating B from the observed information matrix (denoting them by $\mathrm{CI_o}$). The average confidence intervals and the coverage percentages (CP) based on 1000 replications are reported. All the results are reported in Table 1.

Some of points are clear from this experiment. Even for small sample sizes, the performance of MLEs are quite satisfactory in terms of biases and MSEs. It is observed that when $m = n$ and m, n increases, then biases and MSEs decrease. So the consistency property of MLEs of R is validated.

Comparing the average confidence lengths and CPs, it is observed that estimating B either by substituting the parameters by their estimates in Theorem 2 or using observed Fisher information matrix leads to very similar estimates. The confidence intervals based on the MLEs work quite well unless the sample size is chosen very small, say (15,15). Now we assume that the common scale parameter σ is known. In this case, the estimates of R is obtained via the MLE and UMVUE methods. Since for the MLE, the exact distribution is also known, therefore, it can be used to construct confidence intervals. We presented the average confidence lengths (denoting them by $\mathrm{ACL_{MLE}}$). Use of the non-informative prior is preferred, that is, the equality $a_1 = a_2 = b_1 = b_2 = 0$

Table 1. Estimation of parameters when σ is unknown from 1000 replications.

(n,m)	(α,β,σ)	$(\hat{\alpha},\hat{\beta},\hat{\sigma})$	Bias(R)	MSE$(\hat{R})$	CI_e	CI_o	CP
(15, 15)	(1, 1, 1)	(1.02, 1.02, 0.96)	0.0029	0.0085	(0.324, 0.682)	(0.324, 0.682)	0.952
	(1.5, 2, 1)	(1.54, 2.07, 0.96)	0.0016	0.0088	(0.397, 0.749)	(0.397, 0.749)	0.946
	(2, 1.5, 0.5)	(2.03, 1.54, 0.48)	0.0001	0.0084	(0.252, 0.605)	(0.252, 0.605)	0.947
(25, 25)	(1, 1, 1)	(1.01, 1.00, 0.98)	0.0025	0.0058	(0.359, 0.636)	(0.359, 0.636)	0.936
	(1.5, 2, 1)	(1.52, 2.04, 0.98)	0.0048	0.0049	(0.440, 0.712)	(0.440, 0.712)	0.940
	(2, 1.5, 0.5)	(2.00, 1.55, 0.49)	0.0061	0.0051	(0.298, 0.572)	(0.298, 0.571)	0.940
(25, 50)	(1, 1, 1)	(1.01, 1.00, 0.98)	0.0064	0.0037	(0.374, 0.614)	(0.374, 0.614)	0.951
	(1.5, 2, 1)	(1.49, 2.00, 0.99)	0.0009	0.0038	(0.453, 0.688)	(0.453, 0.688)	0.947
	(2, 1.5, 0.5)	(2.04, 1.49, 0.49)	0.0069	0.0035	(0.304, 0.539)	(0.304, 0.539)	0.946
(50, 50)	(1, 1, 1)	(1.00, 1.01, 0.99)	0.0032	0.0028	(0.405, 0.601)	(0.405, 0.601)	0.931
	(1.5, 2, 1)	(1.51, 2.00, 0.99)	0.0010	0.0027	(0.474, 0.667)	(0.474, 0.667)	0.936
	(2, 1.5, 0.5)	(2.00, 1.51, 0.49)	0.0014	0.0025	(0.333, 0.526)	(0.334, 0.526)	0.953

Table 2. Estimation of parameters when σ is known from 1000 replications.

(n,m)	(α,β)	$(\hat{\alpha},\hat{\beta})$	$\hat{R}$	ACL_{MLE}	CP_{MLE}	$\tilde{R}$	$\hat{R}_{Bayes}$	ACL_{Bayes}	CP_{Bayes}
(15, 15)	(1, 1)	(1.02, 1.02)	0.498	0.339	0.949	0.498	0.498	0.324	0.961
	(1.5, 2)	(1.55, 2.05)	0.570	0.333	0.957	0.572	0.568	0.318	0.928
	(2, 1.5)	(2.01, 1.52)	0.434	0.334	0.941	0.432	0.436	0.319	0.962
(25, 25)	(1, 1)	(0.99, 1.01)	0.502	0.268	0.946	0.502	0.502	0.255	0.958
	(1.5, 2)	(1.50, 2.02)	0.574	0.263	0.964	0.575	0.572	0.242	0.966
	(2, 1.5)	(2.04, 1.49)	0.427	0.263	0.951	0.426	0.429	0.247	0.959
(25, 50)	(1, 1)	(1.01, 1.00)	0.499	0.235	0.959	0.501	0.501	0.219	0.959
	(1.5, 2)	(1.52, 2.02)	0.569	0.230	0.951	0.573	0.571	0.214	0.981
	(2, 1.5)	(2.04, 1.51)	0.427	0.230	0.946	0.428	0.430	0.216	0.972
(50, 50)	(1, 1)	(0.99, 1.01)	0.501	0.193	0.955	0.501	0.501	0.177	0.956
	(1.5, 2)	(1.50, 2.02)	0.572	0.189	0.956	0.572	0.571	0.174	0.960
	(2, 1.5)	(2.01, 1.52)	0.430	0.189	0.943	0.429	0.431	0.175	0.955

provides prior distributions which are not proper. We adopt the suggestion of Congdon [20, p. 20] and Kundu and Gupta [21], that is, $a_1 = a_2 = b_1 = b_2 = 0.0001$. The average estimation of Bayes estimates, $\hat{R}_{Bayes}$, is reported under the same prior distributions and based on 1000 replications. Also, we reported the average credible lengths (denoting by ACL_{Bayes}). The results are presented in Table 2. Comparing the average confidence lengths with the average credible lengths, we observe almost the average credible lengths is less than the average credible lengths. In this case as expected, for all the methods as m and n increase, the average biases decrease.

In the third case where the scale parameters σ_1 and σ_2 are different and unknown, first σ_1 and σ_2 are estimated using Equations (10) and (11), respectively. Then, α and β are estimated using Equations (8) and (9), respectively. Finally, the MLE of R is obtained using Equation (12). All the results are reported in Table 3.

7. Date analysis

In this section, a data analysis of the strength data reported by Badar and Priest [22] is presented. Kundu and Gupta [8] observed that Weibull distribution works quite well for these strength data

Table 3. Estimation of parameters when σ_1 and σ_2 are different from 1000 replications.

(n, m)	$(\alpha, \beta, \sigma_1, \sigma_2)$	$(\hat{\alpha}, \hat{\beta}, \hat{\sigma}_1, \hat{\sigma}_2)$	R	$\hat{R}$	Bias(R)	MSE(R)
(25, 25)	(1, 1.5, 1, 0.5)	(0.96, 1.54, 0.96, 0.47)	0.567	0.583	0.0017	0.0067
	(1, 1.5, 1, 2)	(0.97, 1.51, 0.94, 1.89)	0.625	0.634	0.0063	0.0070
	(1, 1.5, 1, 3)	(0.99, 1.50, 0.98, 2.87)	0.638	0.637	0.0022	0.0055
(25, 50)	(1, 1.5, 1, 0.5)	(0.97, 1.49, 0.96, 0.48)	0.561	0.573	0.0091	0.0055
	(1, 1.5, 1, 2)	(1.01, 1.48, 0.95, 1.99)	0.628	0.622	0.0057	0.0036
	(1, 1.5, 1, 3)	(0.97, 1.55, 0.96, 2.96)	0.640	0.652	0.0080	0.0032
(50, 50)	(1, 1.5, 1, 0.5)	(0.99, 1.54, 1.00, 0.48)	0.566	0.569	0.0023	0.0027
	(1, 1.5, 1, 2)	(1.02, 1.51, 0.99, 1.92)	0.624	0.626	0.0036	0.0028
	(1, 1.5, 1, 3)	(1.00, 1.48, 0.99, 2.92)	0.635	0.630	0.0013	0.0032

Table 4. The real data sets.

Data set I (x)							Data set II (y)							
1.901	2.132	2.203	2.228	2.257	2.350	2.361	1.312	1.314	1.479	1.552	1.700	1.803	1.861
2.396	2.397	2.445	2.454	2.474	2.518	2.522	1.865	1.944	1.958	1.966	1.997	2.006	2.021
2.525	2.532	2.575	2.614	2.616	2.618	2.624	2.027	2.055	2.063	2.098	2.140	2.179	2.224
2.659	2.675	2.738	2.740	2.856	2.917	2.928	2.240	2.253	2.270	2.272	2.274	2.301	2.301
2.937	2.937	2.977	2.996	3.030	3.125	3.139	2.359	2.382	2.382	2.426	2.434	2.435	2.478
3.145	3.220	3.223	3.235	3.243	3.264	3.272	2.490	2.511	2.514	2.535	2.554	2.566	2.570
3.294	3.332	3.346	3.377	3.408	3.435	3.493	2.586	2.629	2.633	2.642	2.648	2.684	2.697
3.501	3.537	3.554	3.562	3.628	3.852	3.871	2.726	2.770	2.773	2.800	2.809	2.818	2.821
3.886	3.971	4.024	4.027	4.225	4.395	5.020	2.848	2.880	2.954	3.012	3.067	3.084	3.090
							3.096	3.128	3.233	3.433	3.585	3.585	

Table 5. Sample median, scale parameter, shape parameter, K–S and p-value of the fitted GL models to data sets.

Data set	Sample median	Scale parameter	Shape parameter	K–S	p-Value
1	2.996	0.3643	1.1240	0.1135	0.3914
2	2.478	0.2745	0.9489	0.0492	0.9962

Table 6. Observed frequencies and expected frequencies for modified data set I when fitting the GL model.

Intervals	Observed frequencies	Expected frequencies	Chi-square
<2.5	12	10.5524	0.7401
2.5–3	20	18.5317	
3–3.5	17	19.9094	
3.5–4	9	9.7914	
>4	5	4.2151	

which are presented in Table 4. The GL distribution models to the two data sets are fitted separately. The estimated scale and shape parameters are proposed assuming the location parameter to be known as the sample median for both the data sets. We also obtain the Kolmogrov–Smirnov (K–S) distance between the empirical distribution functions and the fitted distributions, and corresponding p values. All the results have been reported in Table 5. For comparison purposes, we also compute the observed and the expected frequencies, the corresponding chi-square values based on the fitted models in Tables 6 and 7. It is clear that the GL model fits quite well into both the data sets. Because the two scale parameters are not very different, assuming the two parameters

Table 7. Observed frequencies and expected frequencies for modified data set II when fitting the GL model.

Intervals	Observed frequencies	Expected frequencies	Chi-square
<1.76	5	2.3900	0.6452
1.76–2.22	15	15.2904	
2.22–2.68	27	26.9100	
2.68–3.14	18	16.0011	
>3.14	4	5.4027	

Table 8. MLE, asymptotic confidence interval and bootstrap confidence interval of R.

<table>
<tr><th>Data set</th><th>Scale parameter</th><th>Shape parameter</th><th>$\hat{R}$</th><th>CI_{MLE}</th><th>$CI_{Boot\text{-}p}$</th><th>$CI_{Boot\text{-}t}$</th></tr>
<tr><td>1</td><td>0.319</td><td>1.0404</td><td rowspan="2">0.4962</td><td rowspan="2">(0.411, 0.582)</td><td rowspan="2">(0.409, 0.582)</td><td rowspan="2">(0.412, 0.580)</td></tr>
<tr><td>2</td><td>0.319</td><td>1.0562</td></tr>
</table>

are equal, we estimate the parameters, and extracted the 95% confidence interval based on the asymptotic distribution of R and 95% percentile bootstrap method [23] and bootstrap-t method [24] confidence intervals. We reported the results in Table 8.

References

[1] Downtown E. The estimation of $P(Y < X)$ in the normal case. Technometrics 1973;15:551–558.
[2] Tong H. A note on the estimation of $P(Y < X)$ in the exponential case. Technometrics 1974;16:625.
[3] Constantine K, Karson M. The estimation of $P(Y < X)$ in gamma case. Comm. Stat. Comput. Simul. 1986;15:365–388.
[4] Krishnamoorthy K, Mukherjee S, Guo H. Inference on reliability in two-parameter exponential stress–strength model. Metrika 2007;65(3):261–273.
[5] Nadarajah S. Reliability for logistic distributions. Elektronnoe Modelirovanie 2004;26(3):65–82.
[6] Nadarajah S. Reliability for Laplace distributions. Math. Probl. Eng. 2004;2004(2):169–183.
[7] Nadarajah S, Kotz S. Reliability for some bivariate exponential distributions. Math. Probl. Eng. 2006;2006:1–14.
[8] Kundu D, Gupta RD. Estimation of $p[Y < X]$ for Weibull distributions. IEEE Trans. Reliab. 2006;55(2):270–280.
[9] Kotz S, Lumelskii Y, Pensky M. The stress–strength model and its generalization. Singapore: Word Scientific Press; 2003.
[10] Zelterman D. Order statistics of the generalized logistic distribution. Comput. Stat. Data Anal. 1989;7:69–77.
[11] George E, Ojo M. On a generalization of the logistic distribution. Ann. Inst. Statist. Math. 1980;32(1):161–169.
[12] Johnson NL, Kotz S. Distribution in statistics: continuous univariate distributions, Vol. 2. New York: John Wiley; 1970.
[13] Zelterman D. Parameter estimation in the generalized logistic distribution. Comput. Stat. Data Anal. 1987;5(3):177–184.
[14] DasGupta A. Asymptotic theory of statistics and probability. New York: Springer; 2008.
[15] Lehmann EL, Casella G. Theory of point estimation. New York: Springer; 1998.
[16] Tong H. On the estimation of $P(Y < X)$ for exponential families. IEEE Trans. Reliab. 1977;26:54–56.
[17] Lindley DV. Approximate Bayes method. Trabajos de Estadistica 1980;3:281–288.
[18] Ahmad KE, Fakhry ME, Jaheen ZF. Empirical Bayes estimation of $P(Y < X)$ and characterizations of the Burr-type X model. J. Statist. Plann. Inference 1997;64:297–308.
[19] Chen MH, Shao QM. Monte Carlo estimation of Bayesian credible and HPD intervals. J. Comput. Graph. Stat. 1999;8:69–92.
[20] Congdon P. Bayesian statistical modeling. New York: Wiley; 2001.
[21] Kundu D, Gupta RD. Estimation of $p[Y < X]$ for generalized exponential distribution. Metrika 2005;61:291–308.
[22] Badar MG, Priest AM. Statistical aspects of fiber and bundle strength in hybrid composites. In: Hayashi T, Kawata K, Umekava S, editors. Progress in science and engineering of composites. Tokyo: Japanese Society for Composite Materials; 1982. p. 1129–1136.
[23] Efron B. The jackknife, the bootstrap and other resampling plans. CBMS-NSF Regional Conference Series in Applied Mathematics, Vol. 38. Philadelphia, PA: SIAM; 1982.
[24] Hall P. Theoretical comparison of bootstrap confidence intervals. Ann. Stat. 1988;16:927–953.