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Article *in* Archiv der Mathematik · March 1993

DOI: 10.1007/BF01198808

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Arch. Math., Vol. 60, 247–251 (1993)

0003-889X/93/6003-0247 $ 2.50/0

The group of automorphisms of non-associative commutative algebras associated with the group of automorphisms of 2-designs

By

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1. Introduction. Let Ω be a finite set with $|\Omega| = v$, and $\mathscr{B}$ is a collection of b subsets of Ω, each of cardinality k; that is, $|\mathscr{B}| = b$ and $|B| = k$ for all $B \in \mathscr{B}$. The pair $D = (\Omega, \mathscr{B})$ is called a 2-design if each pair of distinct elements of Ω occurs in exactly one element of $\mathscr{B}$. Elements of Ω and $\mathscr{B}$ are called points and blocks, respectively. It is proved that each point occurs in the same, call r, blocks. The constants v, b, k and r are called the parameters of D and it is proved that they satisfy in the following relations:

$$bk = vr, \quad r(k-1) = v - 1.$$

An automorphism of D is defined as a permutation on Ω which induce a permutation on $\mathscr{B}$, and the group of automorphisms of D is denoted by $\operatorname{Aut}(D)$.

Let $D = (\Omega, \mathscr{B})$ be a 2-design with parameters v, b, k and r, $k \geqq 3$. Put $\Omega = \{0, 1, \ldots, n\}$. Let $i, j \in \Omega$ with $i \neq j$, assume that G is a subgroup of $\operatorname{Aut}(D)$ acting 2-transitively on Ω and suppose that $\{i,j\}$, $B(i,j) - \{i,j\}$ and $\Omega - B(i,j)$ are the orbits of $\hat{G}_{i,j}$(the global stabilizer of $\{i,j\}$ in G) on Ω, where $B(i,j)$ denotes the unique block containing i and j. Let A be the commutative (non-associative) G-invariant algebra over a field K of characteristic $\neq 2$. Then by determination of the general structure of commutative (non-associative) algebras associated with doubly transitive permutation groups G by Harada, Theorem 2.4 of [3], A has basis $\{x_1, x_2, \ldots, x_n\}$ and

$$x_i x_i = \dot{a} x_i, \quad 0 \leqq i \leqq n,$$
$$x_i x_j = (\dot{b} - \dot{c})(x_i + x_j) + \dot{c} \sum_{t \in B(i,j)} x_t, \quad 0 \leqq i < j \leqq n,$$

where $\dot{a}, \dot{b}, \dot{c} \in K$, $\dot{a} + (n-1)\dot{b} = (k-2)\dot{c}$ and $x_0 = -x_1 - x_2 - \ldots - x_n$. (The product with $i = 0$ is a consequence of the others.)

Let G^* be a group with $PSL(m, q) \leqq G^* \leqq P\Gamma L(m, q)$, $m \geqq 3$, where $PSL(m, q)$ and $P\Gamma L(m, q)$ are the projective special linear group and projective general semilinear group, respectively. It is proved that $PSL(m, q)$, and hence G^*, acts 2-transitively on Ω, the points of $(m-1)$-dimensional projective space $PG(m-1, q)$, associated with $V(m, q)$, an m-dimensional vector space over a finite field having q-elements.

Let A^* be the commutative (non-associative) G^*-invariant algebra. By Fundamental Theorem of Projective Geometry we have $P\Gamma L(m, q) \cong \mathrm{Aut}(PG(m-1, q)) \cong \mathrm{Aut}(P_1(m-1, q))$, where $P_1(m-1, q)$ is a special 2-design whose points are the points of $PG(m-1, q)$ and its blocks are the (projective) lines of $PG(m-1, q)$. Therefore, the algebra A^* is a special case of the algebra A, constructed before.

Harada [4] investigated the property of the regular representation ϕ of A^*. In particular, he determined the minimal polynomial of ϕ for all basis elements of A^*, namely $x_0, x_1, \ldots, x_n$. Then Darafsheh [2] generalized some of the results of Harada to the algebra A.

A question raised by Harada is whether every doubly transitive permutation Group G affords some commutative G-invariant algebra A for which $\mathrm{Aut}(A) \cong G$. So far no one could answer this question, however, Allen [1] answered affirmatively it in some cases of algebra A^*.

Using the results of Harada, Theorem 4.1 of [4], and Allen, Theorem 2 of [1], Narang [5] or [6] in his Ph. D. thesis generalized some results of the Harada.

2. Generalities.

Theorem 2.1. *Let A be a commutative G-invariant algebra defined in the introduction with $C \neq 0$, associative with $G \leqq \mathrm{Aut}(D)$. Then the subgroup of $\mathrm{Aut}(A)$ permuting the elements $\{x_i \mid 0 \leqq i \leqq n\}$ among themselves is equal to $\mathrm{Aut}(D)$.*

Proof. Let $\sigma \in \mathrm{Aut}(D)$. Then by a theorem in elementary linear algebra there is a unique linear operator, say $\tilde{\sigma}$, on A such that

$$(1) \qquad (x_i)^{\tilde{\sigma}} = x_i\sigma, \quad 0 \leqq i \leqq n.$$

The relation (1) shows that $\tilde{\sigma}$ acts on the set $\{x_0, x_1, \ldots, x_n\}$, so it may be thought as an element of S_{n+1}. We next prove that $\tilde{\sigma} \in \mathrm{Aut}(A)$. To do this, we have

$$\begin{aligned}
(x_i x_i)^{\tilde{\sigma}} &= (\dot{a} x_i)^{\tilde{\sigma}} = \dot{a}(x_i)^{\tilde{\sigma}} = \dot{a} x_i \sigma = x_i \sigma x_i \sigma \\
&= (x_i)^{\tilde{\sigma}} (x_i)^{\tilde{\sigma}}, \quad \text{for } 0 \leqq i \leqq n, \quad \text{and} \\
(x_i x_j)^{\tilde{\sigma}} &= [(\dot{b} - \dot{c})(x_i + x_j) + \dot{c} \sum_{t \in B(i,j)} x_t]^{\tilde{\sigma}} \\
&= (\dot{b} - \dot{c})(x_{i^\sigma} + x_{j^\sigma}) + \dot{c} \sum_{t \in B(i,j)} x_{t^\sigma} \\
&= (\dot{b} - \dot{c})(x_{i^\sigma} + x_{j^\sigma}) + \dot{c} \sum_{t^\sigma \in B(i^\sigma, j^\sigma)} x_{t^\sigma} \,(\text{since } \sigma \in \mathrm{Aut}(D)) \\
&= (x_{i^\sigma} x_{j^\sigma}) = (x_i)^{\tilde{\sigma}} (x_j)^{\tilde{\sigma}}, \quad \text{for } 0 \leqq i < j \leqq n.
\end{aligned}$$

Therefore, $\tilde{\sigma} \in \mathrm{Aut}(A)$. Thus each element of $\mathrm{Aut}(D)$ induces and element of $\mathrm{Aut}(A) \cap S_{n+1}$.

Conversely, we will show that all elements of $\mathrm{Aut}(A) \cap S_{n+1}$ are obtained in this way. Let $\tilde{\sigma} \in \mathrm{Aut}(A) \cap S_{n+1}$. Then $(x_i)^{\tilde{\sigma}} = x_{i^\sigma}$, where σ^{-1} is a permutation in S_{n+1} inducing $\tilde{\sigma}$. We shall show that $\sigma \in \mathrm{Aut}(D)$. Since σ is a permutation on Ω, it is enough to show that σ is a permutation on $\mathscr{B}$. Let i and j be distinct elements of Ω. Since $\tilde{\sigma} \in \mathrm{Aut}(A)$, $(x_i x_j)^{\tilde{\sigma}} = (x_i)^{\tilde{\sigma}} (x_j)^{\tilde{\sigma}}$. But

$$(x_i)^{\tilde{\sigma}} (x_j)^{\tilde{\sigma}} = x_{i^\sigma} x_{j^\sigma} = (\dot{b} - \dot{c})(x_{i^\sigma} + x_{j^\sigma}) + \dot{c} \sum_{t \in B(i^\sigma, j^\sigma)} x_t, \tag{2}$$

and

$$(x_i x_j)^{\tilde{\sigma}} = (\dot{b} - \dot{c})(x_{i^\sigma} + x_{j^\sigma}) + \dot{c} \sum_{t \in B(i,j)} x_{t^\sigma}. \tag{3}$$

We see that when t varies the whole of $B(i,j)$, t^σ varies the whole of $B(i,j)^\sigma$, and vice versa. So (3) refers to

$$(x_i x_j)^{\tilde{\sigma}} = (\dot{b} - \dot{c})(x_{i^\sigma} + x_{j^\sigma}) + \dot{c} \sum_{t \in B(i,j)} \sigma^x t. \tag{4}$$

Since $\dot{c} \neq 0$ by assumption, then from (2) and (4) we get

$$\sum_{t \in B(i^\sigma, j^\sigma)} x_t = \sum_{t \in B(i,j)^\sigma} x_t \tag{5}$$

for all $0 \leqq i \neq j \leqq n$. If $B(i^\sigma, j^\sigma) \neq B(i,j)^\sigma$ for some $0 \leqq i \neq j \leqq n$, then (5) gives a linear combination of x_i's, $0 \leqq i \leqq n$, not all of its coefficients are either zero or non-zero, equal with zero. Replacing x_0 by $-x_1 - x_2 - \ldots - x_n$, we will see that $\{x_1, x_2, \ldots, x_n\}$ is a lineary dependent set, contradiction. Therefore, $B(i^\sigma, j^\sigma) = B(i,j)^\sigma$ for all $0 \leqq i \neq j \leqq n$, that shows σ is a permutation on $\mathscr{B}$, as desired.

Theorem 2.2. *Let $D = (\Omega, \mathscr{B})$ be a 2-design with parameters v, b, k and r, $k \geqq 3$, and suppose that $G \leqq \mathrm{Aut}(D)$ be as before. Let A be the commutative algebra over a field $K(\mathrm{char}(K) \neq 2)$ with basis $\{x_1, x_2, \ldots, x_n\}$, associated with G. Then $\mathrm{Aut}(A) \cong \mathrm{Aut}(D)$, provided one of the followings holds:*

(i) $\dot{b} = \dot{c}$, $\dot{c} \neq 0$, *and in* K, $n + 1 \neq 0$, $k - 2 \neq 0$, $k \neq 0$; *or*
(ii) $\dot{b} = \dot{a} + \dot{c}$, $\dot{b} \neq 0$, $\dot{c} \neq 0$, *and* $\mathrm{char}(K) > r - 1$ *or* $\mathrm{char}(K) \neq 0$, *and* $n + 1 \neq 0$ *in* K.

Note. It is proved that the number r of blocks of the 2-design $P_1(m-1, q)$ containing one (each) point is $(q^{m-1} - 1)/(q - 1)$.

Proof. The proof in case (i) is similar to the proof of Theorem 1.3 (i) by Narang [5] or [6] (it is enough to interchange q by $k - 1$). Also, by replacing q by $k - 1$, the case (ii) is proved as similar as Theorem 1.3 (ii) [5]. It must only be noted that

(a) $k - 1 \neq 0$ and $n + 1 - \dot{c} \neq 0$ in K,
(b) $n\dot{c} \neq (n - \dot{c} + 1)$ in Claim of Subcase (ii″) of case (ii) in Lemma 5.3.19 of [5] or Lemma 3.18 of [6].

In fact, we may assume $\dot{b} = 1$, such as Narang. From $\dot{b} = \dot{a} + \dot{c}$, we get $\dot{a} = 1 - \dot{c}$ and it follows from $\dot{a} + (n-1)\dot{b} = (k-2)\dot{c}$ that

$$\dot{c} = \frac{n}{k-1} = \frac{v-1}{k-1} = r,$$

since $n+1=|\Omega|=v$ and $r(k-1)=v-1$. Now $r>k$, otherwise, if $r \leqq k$, then we have

$$\frac{v-1}{k-1} \leqq k, \quad \frac{v-1}{k(k-1)} \leqq 1, \quad \frac{v(v-1)}{k(k-1)} \leqq v, \quad b \leqq v, \tag{1}$$

since $bk = vr$, whence $b = \frac{v}{k} r = \frac{v(v-1)}{k(k-1)}$. But the last inequality in (1) contradict the Fischer's inequality; $v \leqq b$. (Note that if $b = v$, then $k = r$). Therefore $r > k$ has to hold, whence $r - 1 > k - 1$. Since $\operatorname{char}(K) > r - 1$ or $\operatorname{char}(K) = 0$, then $k - 1 \neq 0$ in K.

By assumption $k \geqq 3$, or $k - 2 \geqq 1$, which gives $r(k-2) \geqq r$. On the other hand we have $r(k-1) = v - 1$, whence $v = r(k-1) + 1$. Now we have

$$v - r = r(k-1) + 1 - r = r(k-2) + 1 \geqq r + 1 > r - 1.$$

Since $\operatorname{char}(K) > r - 1$ or $\operatorname{char}(K) = 0$, thus $v - r \neq 0$ in K, or $n + 1 - \dot{c} \neq 0$ in K, which completes our first conclusion (a).

For (b) we show equivalently that $(v-1)r \neq (v-r)^2$. Suppose that $(v-1)r = (v-r)^2$. Then we have

$$(v-1)\frac{v-1}{k-1} = \left(v - \frac{v-1}{k-1}\right)^2 = \left(\frac{v(k-1)-(v-1)}{k-1}\right)^2 = \left(\frac{vk-2v+1}{k-1}\right)^2$$
$$< \left(\frac{v^2-2v+1}{k-1}\right)^2 \quad (\text{since } 0 \leqq k < v),$$

whence

$$\frac{(v-1)^2}{k-1} < \left(\frac{(v-1)^2}{k-1}\right)^2. \tag{2}$$

But $\frac{(v-1)^2}{k-1}$ is a positive integer, so (2) is impossible. This shows that $(v-1)r \neq (v-r)^2$, as desired.

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Eingegangen am 8. 1. 1992

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